

Combinatorial quantum field theory

Assignment 1

due by October 21

Michael Borinsky

You must write your solutions on your own. If you discussed any of the problems with other students, clearly state this at the beginning of your solution to each problem and list the names of those students. Email your solution to me as a PDF file (LaTeX is preferred) Email: mborinsky@perimeterinstitute.ca.

1. **18 marks** The Euler- Γ function generalizes the factorial function to non-integer values. We have $n! = \Gamma(n+1)$ for all $n \geq 0$. One way to define Γ is via the integral expression

$$\Gamma(z) = \int_0^\infty t^z e^{-t} \frac{dt}{t} \quad \text{for all } z > 0.$$

- (a) Bring the integral expression for $\Gamma(z)$ above into a form amenable to Theorem 13. (Hint: Change the integration variable to x via $t = ze^x$).
- (b) Apply Theorem 13 to prove a sum-over-graphs formula for the coefficients c_k of the following $z \rightarrow \infty$ asymptotic expansion,

$$\frac{\Gamma(z)}{\sqrt{\frac{2\pi}{z}} e^{-z} z^z} \sim \sum_{k \geq 0} c_k z^{-k}.$$

This asymptotic expansion is essential in physics and in many mathematical fields; for instance, in combinatorics due to the connection to the factorial.

- (c) Compute the first two coefficients (i.e. c_0 and c_1) of the expansion using graphs.
- (d) Use the following formula, the *Stirling asymptotic expansion* of Γ , to find a more compact expression for the coefficients c_k above. For all $R \geq 0$,

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log(2\pi) + \sum_{k=1}^{R-1} \frac{B_{k+1}}{k(k+1)} z^{-k} + \mathcal{O}(z^{-R}),$$

where B_k are the *Bernoulli numbers*.

- (e) Use graphs to compute the value of the second Bernoulli number B_2 .
- (f) The Bernoulli numbers B_k vanish if k is odd and larger than 1. Use this to prove a vanishing statement for an alternating sum over connected graphs of fixed Euler characteristic.

2. **6 marks** Let $p_1, p_2, \dots$ be an infinite set of variables. For a permutation $\alpha \in \text{Sym}(H)$ with c_1^α 1-cycles, c_2^α 2-cycles, etc, we define the monomial $p^\alpha = \prod_{k \geq 0} p_k^{c_k^\alpha}$. For a given H -labeled graph G , the polynomial

$$C(G) = \frac{1}{|\text{Aut}(G)|} \sum_{\alpha \in \text{Aut}(G)} p^\alpha,$$

is the *character* of the representation of $\text{Sym}(H)$ associated with G . It is also known as the *Pólya cycle-index polynomial*.

- (a) Compute $C(\infty)$ and $C(\text{graph with 3 vertices and 2 edges})$.
- (b) Observe that if $p_k = 1$ for all $k \geq 1$, then $C(G)$ is 1. Prove that $C(G) \in \mathbb{Z}$ if $p_k = -1$ for all $k \geq 1$.
3. **10 marks** Let $f(x)$ and $g(x)$ be power series $f(x) = 1 + \sum_{n \geq 1} f_n x^n$, and $g(x) = \sum_{n \geq 1} g_n x^n$ in $\mathbb{Q}[[x]]$, related by $f(x) = \exp(g(x))$.
- (a) Prove a recursion that computes f_n when g_n is known. Also, prove a recursion that computes g_n when f_n is known.
- (b) Prove: If $g_n > 0$, then $f_n > 0$. Also prove: If $f_n < 0$, then $g_n < 0$.
- (c) Find a graph-based expression for the $z \rightarrow \infty$ asymptotic expansion of

$$I(z) = \sqrt{\frac{z}{2\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp\left(-z \frac{\sin^2(x)}{2}\right) dx$$

- (d) Compute the asymptotic expansion combinatorially, neglecting all terms in $\mathcal{O}(z^{-2})$.
- (e) Prove a closed-form expression for the coefficients of this asymptotic expansion.
4. **6 marks** For a given H -labeled graph $G = (V, E)$, define $\mathbb{Q}H, \mathbb{Q}V, \mathbb{Q}E$ that are generated by the respective sets. Consider the linear map $\partial : \mathbb{Q}H \rightarrow \mathbb{Q}V \oplus \mathbb{Q}E, h \mapsto v_h - e_h$ that maps a half-edge generator to the difference of the generators for the vertex and edge v_h, e_h to which h belongs.
- (a) Prove that $\dim \ker \partial = \#C(G) - \chi(G)$, where $\#C(G)$ is the number of connected components of G and $\chi(G) = |V_G| - |E_G|$ is the Euler characteristic. (Hint: Give a combinatorial interpretation to the cokernel of ∂ .)
- (b) Let $G = (V, E)$ be an H -labeled tree. For a given automorphism $\alpha \in \text{Aut}(G)$, let α_V and α_E be the permutations that α induces on the sets V and E . Show that $\text{sign}(\alpha) = \text{sign}(\alpha_E) \text{sign}(\alpha_V)$.