

Combinatorial quantum field theory

Assignment 2

due by November 18

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You must write your solutions on your own. If you discussed any of the problems with other students, clearly state this at the beginning of your solution to each problem and list the names of those students. Email your solution to me as a PDF file (LaTeX is preferred) Email: mborinsky@perimeterinstitute.ca.

1. **10 marks** Let

$$A_n = \sum_{\substack{G \in \mathcal{G}^n \\ \deg \geq 3 \\ \chi(G) = -n}} \frac{1}{|\text{Aut}(G)|},$$

where the sum is over all isomorphism classes of graphs with Euler characteristic $-n$ without vertices of degree 0, 1, 2 (and no legs). Compute the first two coefficients of the $n \rightarrow \infty$ asymptotic expansion of A_n .

Hint: Use the abbreviation $\tau \approx 1.25643\dots$ for the unique positive solution of $2\tau = e^\tau - 1$.

2. **8 marks**

(a) For $\sigma_v, \sigma_w \in \{-1, +1\}$, prove the formula

$$\exp(\beta J \sigma_v \sigma_w) = \rho(1 + \kappa \sigma_v \sigma_w),$$

where $\rho = \cosh(\beta J)$ and $\kappa = \tanh(\beta J)$.

(b) Compute the partition function of the Ising model for the $\textcircled{D}$ and the $\textcircled{O}\textcircled{O}$ graph. For one of the two graphs, do the computation both by summing over all vertex configurations and by summing over all even subgraphs.

3. **8 marks** Let $\alpha, \beta > 0$, $\Gamma(z) = \int_0^\infty t^z e^{-t} \frac{dt}{t}$, and $\Gamma_\beta^\alpha(z) = \alpha^{-z-\beta} \Gamma(z + \beta)$. Prove that for all $R \geq 0$,

$$\sum_{k=R}^{n-R} \Gamma_\beta^\alpha(k) \Gamma_\beta^\alpha(n-k) \in \mathcal{O}(\Gamma_\beta^\alpha(n-R)).$$

Hint: Use that $\Gamma(z)$ is log-convex and that $z\Gamma(z) = \Gamma(z+1)$ for all $z > 0$.

4. **14 marks** The formula for the partition function of the Ising model of a graph G is

$$Z_G(\beta, J) = \sum_{\sigma \in \{-1, +1\}^{V_G}} \exp \left(\beta J \sum_{\substack{e \in E_G \\ e \cong (u, v)}} \sigma_u \sigma_v \right).$$

The partition function of the Ising model *on random 3-regular graphs* is the power series in z^{-1} that sums $Z_G(\beta, J)$ over all graphs:

$$Z(z, \beta, J) = \sum_G \frac{Z_G(\beta, J)}{|\text{Aut}(G)|} z^{\chi(G)},$$

where the sum is over all 3-regular unlabeled, possibly disconnected graphs G without legs.

- (a) Show that $Z(z, \beta, J)$ is a power series in $\mathbb{Q}[\kappa][[\rho^3 z^{-1}]]$ with $\rho = \cosh(\beta J)$ and $\kappa = \tanh(\beta J)$. (As always, you can refer to arguments from the lecture and you do not have to replicate them.)
- (b) Compute $Z(z, \beta, J)$ ignoring all terms in $\mathcal{O}(z^{-2})$ for $z \rightarrow \infty$.
- (c) Briefly explain why the following generating function identity holds,

$$\sum_G \frac{x^{n(G)}}{|\text{Aut}(G)|} = \frac{1}{2} \log \frac{1}{1-x},$$

where the sum is over all connected, 3-regular, bridgeless unlabeled graphs of Euler characteristic 0 with arbitrarily many legs and $n(G)$ is the number of legs of G . Recall that the empty graph is not connected. A graph is bridgeless if it has no edge whose removal increases the number of connected components.

- (d) Use the subgraph-sum variant of Theorem 12 (Theorem 14) to prove that

$$Z(z, \beta, J) = \sum_{s \geq 0} (z^{-1} \rho^3)^s (2s-1)!! [x^{2s}] \exp \left(2z \rho^{-3} \frac{x^3}{3!} + \frac{1}{2} \log \frac{1}{1-2\kappa x} \right).$$

- (e) **Bonus question (not graded):** Solve the Ising model on a random 3-regular graph. To do so, compute the $n \rightarrow \infty$ asymptotic behaviour of the coefficients $A_n(\kappa)$ in $Z(z, \beta, J) = \sum_{n \geq 0} A_n(\kappa) (\rho^3 z^{-1})^n$. Hint: Write $Z(z, \beta, J)$ as an integral expression. Transform this integral into an integral over the exponential of a polynomial. Study the locations of the critical points of this polynomial. For which values of κ do you expect the $n \rightarrow \infty$ asymptotic behaviour of $A_n(\kappa)$ to change non-analytically?