# Combinatorial quantum field theory
# classroom exercise session 1

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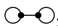
1. Prove the equivalence of the topological and the combinatorial graph definitions from the lecture.

2. (a) Find all half-edge/$\{a, b, c, d\}$-labeled representatives of the graph .
   (b) Find generators of $\mathrm{Aut}(G)$ for half-edge labeled representatives of the graphs , , , .
   (c) Compute the orders of the automorphism group of each of the above graphs.

3. (a) For any half-edge labeled graph, prove that $\sum_{v \in V_G} |v| = 2|E_G|$.
   (b) The Euler characteristic $\chi(G) = |V_G| - |E_G|$ is an (important) homotopy invariant of $G$. A graph is *admissible* if it has no vertices of degree $0, 1$ or $2$. Show that $\chi(G) < 0$ for all non-empty admissible graphs $G$.
   (c) Show that the number of admissible graphs with fixed Euler characteristic is finite.

4. (a) Compute the bivariate generating function $\sum_{n,k \geq 0} \binom{n}{k} x^k y^n$.
   (b) Find a recursion equation for the sum $f_n = \sum_{k=0}^{\lceil n/2 \rceil} \binom{n-k}{k}$ defined for $n \geq 0$. (Use a)

5. (a) For a given $H$-labeled graph $G = (V, E)$, define $\mathbb{Q}$-vector spaces $\mathbb{Q}H, \mathbb{Q}V, \mathbb{Q}E$ that are generated by the respective sets. Consider the linear map $\partial : \mathbb{Q}H \to \mathbb{Q}V \oplus \mathbb{Q}E, h \mapsto v_h - e_h$ that maps a half-edge generator to the difference of the generators for the vertex and edge $v_h, e_h$ to which $h$ belongs.

      Prove that $\dim \ker \partial = \#C(G) - \chi(G)$, where $\#C(G)$ is the number of connected components of $G$.
   (b) Let $G = (V, E)$ be an $H$-labeled tree. For a given automorphism $\alpha \in \mathrm{Aut}(G)$, let $\alpha_V$ and $\alpha_E$ be the permutations that $\alpha$ induces on the sets $V$ and $E$. Show that $\mathrm{sign}(\alpha) = \mathrm{sign}(\alpha_E)\,\mathrm{sign}(\alpha_V)$.