Combinatorial quantum field theory classroom exercise session 2

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1. (a) Prove directly (without the formula from the lecture) that the number of matchings (set partitions into blocks of size two) of a set of cardinality $2s$ is $(2s-1)!! = (2s-1)\cdot(2s-3)\cdots 3\cdot 1$.

   (b) Prove that for all $z > 0$, (Hint: Use $\int_{\mathbb{R}} e^{-x^2}\mathrm{d}x = \sqrt{\pi}$),

   $$\sqrt{\frac{z}{2\pi}}\int_{-\infty}^{\infty} x^{2n+1} e^{-z\frac{x^2}{2}}\mathrm{d}x = 0.$$

   and

   $$\sqrt{\frac{z}{2\pi}}\int_{-\infty}^{\infty} x^{2n} e^{-z\frac{x^2}{2}}\mathrm{d}x = z^{-n}(2n-1)!!$$

2. (a) Use the theorem from the lecture to derive a formula for the number of $\{1,\ldots,2s\}$-labeled k-regular graphs. (All vertices have degree k.)

   (b) Find one representative for each isomorphism class of 3-regular graphs with two vertices and for each isomorphism class of 4-regular graphs with two vertices.

   (c) Compute the cardinality of the automorphism group of each graph from (b).

   (d) Certify that you found all graphs in (b) using Theorem 6 and the result of (a).

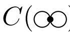
3. (a) Let $f(x)$ and $g(x)$ be power series $f(x) = 1 + \sum_{n\geq 1} f_n x^n$, and $g(x) = \sum_{n\geq 1} g_n x^n$ in $\mathbb{Q}[[x]]$, related by $f(x) = \exp(g(x))$. Find a recursion equation that computes f_n when g_n is known and a recursion that computes g_n when f_n is known.

   (b) Prove: If $g_n > 0$, then $f_n > 0$. And, if $f_n < 0$, then $g_n < 0$.

4. Let $p_1, p_2, \ldots$ be an infinite set of variables. For a permutation $\alpha \in \mathrm{Sym}(H)$ with c_1^α 1-cycles, c_2^α 2-cycles, etc, we define the monomial $p^\alpha = \prod_{k\geq 0} p_k^{c_k^\alpha}$. For a given H-labeled graph G, the polynomial

   $$C(G) = \frac{1}{|\operatorname{Aut}(G)|}\sum_{\alpha\in\operatorname{Aut}(G)} p^\alpha,$$

   is the *character* of the representation of $\mathrm{Sym}(G)$ associated with G. Historically, it is also known as the *Pólya cycle-index polynomial*.

   (a) Compute $C($$)$.

   (b) Prove that $C(G)$ is an integer for all graph G and for all $k \geq 1$ we have $p_k = -1$.

5. (a) Compute the bivariate generating function $\sum_{n,k\geq 0}\binom{n}{k}x^k y^n$.

   (b) Find a recursion equation for the sum $f_n = \sum_{k=0}^{\lceil n/2\rceil}\binom{n-k}{k}$ defined for $n \geq 0$. (Use a)