Combinatorial quantum field theory classroom exercise session 3

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1. Let $\varepsilon(z) = z^{-\frac{5}{12}}$, and $k \in \mathbb{Z}_{\geq 0}$, prove that in the limit $z \to \infty$,
$$\int_{\varepsilon(z)}^{\infty} x^k e^{-zx^2/2} \mathrm{d}x \in \mathcal{O}(z^{-R}) \text{ for all } R \geq 0.$$

2. (a) Use Theorem 12 to find an expression for the generating function of 2-regular graphs.
   (b) Before and after a variable transformation, interpret the integral
   $$I(z) = \sqrt{\frac{z}{2\pi}} \int_{-\infty}^{\infty} \exp\left(z\left(-\frac{x^2}{2} + \lambda_2 \frac{x^2}{2}\right)\right) \mathrm{d}x\,.$$
   as in the setup of Theorem 13.
   (c) Give a generating function interpretation of $I(z)$.

3. Let $f(z) = \int_{\mathbb{R}} \frac{e^{-x^2/2}}{1+x^2/z} \frac{\mathrm{d}x}{\sqrt{2\pi}}$, prove that for all $R \geq 0$, $f(z) = \sum_{k=0}^{R-1} z^{-k}(2k-1)!! + \mathcal{O}(z^{-R})$.

4. (a) Find a graph-based expression for the $z \to \infty$ asymptotic expansion of
   $$I(z) = \sqrt{\frac{z}{2\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp\left(-z\frac{\sin^2(x)}{2}\right) \mathrm{d}x$$
   (b) Compute the asymptotic expansion graphically neglecting all terms in $\mathcal{O}(z^{-2})$.
   (c) Find a closed-form expression for the coefficients of this asymptotic expansion.

5. (a) Let $f(x)$ and $g(x)$ be power series $f(x) = 1 + \sum_{n\geq 1} f_n x^n$, and $g(x) = \sum_{n\geq 1} g_n x^n$ in $\mathbb{Q}[[x]]$, related by $f(x) = \exp(g(x))$. Find a recursion equation that computes f_n when g_n is known and a recursion that computes g_n when f_n is known.
   (b) Prove: If $g_n > 0$, then $f_n > 0$. And, if $f_n < 0$, then $g_n < 0$.