Combinatorial quantum field theory
classroom exercise session 4

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1. Let $A(\lambda, j) = \sum_{s \geq 0} (2s-1)!![x^{2s}] \exp(\lambda \frac{x^4}{4!} + jx)$.
   (a) Give a combinatorial interpretation for the coefficients of $A(\lambda, j) = \sum_{k,\ell} a_{k,\ell} \lambda^k j^\ell$.
   (b) Show that A fulfills the PDE:
   $$\frac{\partial A}{\partial \lambda} = \frac{1}{4!} \frac{\partial^4 A}{\partial j^4}$$
   (c) Translate this PDE into a relation among the coefficients $a_{k,\ell}$ of $A(\lambda, j)$.
   (d) Find a combinatorial interpretation for this relation.
   (e) Find a PDE fulfilled by $\log A(\lambda, j)$ and a non-linear recursion equation for the coefficients of $\log A(\lambda, j) = \sum_{k,\ell} b_{k,\ell} \lambda^k j^\ell$.
   (f) Interpret this new recursion equation combinatorially.

2. (a) For $\sigma_v, \sigma_w \in \{-1, +1\}$, prove the formula
   $$\exp(\beta J \sigma_v \sigma_w) = \rho(1 + \kappa \sigma_v \sigma_w),$$
   where $\rho = \cosh(\beta J)$ and $\kappa = \tanh(\beta J)$.
   (b) Compute the partition function of the Ising model for the  and the  graph. Either do the computation by summing over all vertex configurations or over all even subgraphs. Crosscheck the results as a team.

3. (a) Compute the bivariate generating function $\sum_{n,k \geq 0} \binom{n}{k} x^k y^n$.
   (b) Find a recursion equation for the sum $f_n = \sum_{k=0}^{\lceil n/2 \rceil} \binom{n-k}{k}$ defined for $n \geq 0$. (Use a)