Combinatorial quantum field theory
Solutions to assignment 1

Michael Borinsky

1. *The Euler-Γ function generalizes the factorial function to non-integer values. We have $n! = \Gamma(n+1)$ for all $n \geq 0$. One way to define Γ is via the integral expression*

$$\Gamma(z) = \int_0^\infty t^z e^{-t} \frac{\mathrm{d}t}{t} \quad \textit{for all} \quad z > 0.$$

   (a) *Bring the integral expression for $\Gamma(z)$ above into a form amenable to Theorem 13. (Hint: Change the integration variable to x via $t = ze^x$).*

   After the substitution, we get,

$$\Gamma(z) = \int_{-\infty}^\infty z^z e^{zx - ze^x} \mathrm{d}x = z^z e^{-z} \int_{-\infty}^\infty e^{z(1+x-e^x)} \mathrm{d}x\,. \tag{1}$$

   Define $g(x) = 1 + x - e^x$. The derivative of g equals, $g'(x) = 1 - e^x$. We find that $g'(0) = 0$, $g'(x) > 0$ for all $x < 0$, and $g'(x) < 0$ for all $x > 0$. It follows that g is strictly increasing for $x < 0$, strictly decreasing for $x > 0$, and that it has a maximum at $x = 0$. This maximum at $x = 0$ is hence the unique global supremum. Further, $g(x)$ is entire. In particular, we can locally expand it around $x = 0$ and we find that $g''(0) = -1$.

   We also need to prove that the integral on the right-hand side of (1) exists for large enough z. We may know that $\Gamma(z)$ is finite for all $z > 0$, so this existence follows.

   Here is an argument that does not use this information:

$$\int_{-\infty}^\infty e^{z(1+x-e^x)} \mathrm{d}x = \int_{-\infty}^0 e^{z(1+x-e^x)} \mathrm{d}x + \int_0^\infty e^{z(1+x-e^x)} \mathrm{d}x$$

   Because $0 \leq e^x \leq 1$ for all $x \leq 0$, we have

$$\int_{-\infty}^0 e^{z(1+x-e^x)} \mathrm{d}x \leq \int_{-\infty}^0 e^{z(1+x)} \mathrm{d}x = \frac{1}{z} e^z\,.$$

   And because $1 + x - e^x = -\sum_{k \geq 2} \frac{x^k}{k!} \leq -\frac{x^2}{2}$ for all $x \geq 0$, we have

$$\int_0^\infty e^{z(1+x-e^x)} \mathrm{d}x \leq \int_0^\infty e^{-z\frac{x^2}{2}} \mathrm{d}x = \frac{1}{2}\sqrt{\frac{2\pi}{z}}\,.$$

   So, $\int_{-\infty}^\infty e^{z(1+x-e^x)} \mathrm{d}x$ is finite for all $z > 0$.

   Hence, the requirements of Theorem 13 are fulfilled.

   (b) *Apply Theorem 13 to prove a sum-over-graphs formula for the coefficients c_k of the following $z \to \infty$ asymptotic expansion,*

$$\frac{\Gamma(z)}{\sqrt{\frac{2\pi}{z}} e^{-z} z^z} \sim \sum_{k \geq 0} c_k z^{-k}.$$

This asymptotic expansion is essential in physics and in many mathematical fields; for instance, in combinatorics due to the connection to the factorial.

We have

$$g(x) = 1 + x - e^x = -\sum_{k\geq 2} \frac{x^k}{k!} = -\frac{x^2}{2} + \sum_{k\geq 3} \lambda_k \frac{x^k}{k!} \, ,$$

where $\lambda_k = -1$ for all $k \geq 3$. Hence, by Theorem 13 and Eq. (1),

$$\frac{\Gamma(z)}{\sqrt{\frac{2\pi}{z}} e^{-z} z^z} = \sqrt{\frac{z}{2\pi}} \int_{-\infty}^{\infty} e^{z(1+x-e^x)} \mathrm{d}x \sim \sum_{k\geq 0} c_k z^{-k} \quad \text{for large } z\text{, where} \tag{2}$$

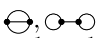
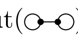
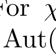
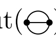
$$c_k = \sum_{\substack{G\in\mathcal{G}^u|_{\deg\geq 3} \\ \chi(G)=-k}} \frac{\prod_{v\in V_G} \lambda_{|v|}}{|\operatorname{Aut}(G)|} = \sum_{\substack{G\in\mathcal{G}^u|_{\deg\geq 3} \\ \chi(G)=-k}} \frac{(-1)^{|V_G|}}{|\operatorname{Aut}(G)|} \, .$$

(c) *Compute the first two coefficients (i.e. c_0 and c_1) of the expansion using graphs.*

We discussed the following in previous exercises, but it is good to recall it:

Because the graphs need to have vertex degree ≥ 3, the number of vertices cannot exceed $2|E_G|/3$. Therefore, $-\chi(G) = |E_G| - |V_G| \geq \frac{1}{3}|E_G| = \frac{1}{3}(|V_G| - \chi(G)) \Rightarrow -\chi(G) \geq \frac{1}{2}|V_G|$. So, if $\chi(G) = 0$, we only need to consider graphs without edges and vertices. For $\chi(G) = -1$, we need to sum all graphs with up to 3 edges and two vertices.

The empty graph $G = \emptyset$ counts as a graph with $\operatorname{Aut}(\emptyset) = 1$, so $c_0 = 1$.

For $\chi(G) = -1$, we must sum over the graphs , and . We find that $|\operatorname{Aut}($$)| = |\operatorname{Aut}($$)| = 8$ and $|\operatorname{Aut}($$)| = 12$. Hence $c_1 = -\frac{1}{8} + \frac{1}{8} + \frac{1}{12} = \frac{1}{12}$. (We could add an argument here, proving that these are all relevant graphs.)

(d) *Use the following formula, the* Stirling asymptotic expansion *of* Γ*, to find a more compact expression for the coefficients* c_k *above. For all* $R \geq 0$,

$$\log\Gamma(z) = (z - \frac{1}{2})\log z - z + \frac{1}{2}\log(2\pi) + \sum_{k=1}^{R-1} \frac{B_{k+1}}{k(k+1)} z^{-k} + \mathcal{O}(z^{-R}),$$

where B_k *are the* Bernoulli numbers.

Let $\beta_k = \frac{B_{k+1}}{k(k+1)}$. We need to combine the formula above with Eq. (2) by matching the coefficients of the asymptotic expansions order by order. After some manipulations, we get the identity

$$c_k = [z^{-k}] \exp\left(\sum_{\ell\geq 1} \beta_\ell z^{-\ell}\right) = [z^{-k}] \prod_{\ell\geq 1} \exp\left(\beta_\ell z^{-\ell}\right)$$

Because of the coefficient extraction operator, we can truncate the product for $\ell > k$. Hence, with $\exp(x) = \sum_{\ell\geq 0} \frac{x^\ell}{\ell!}$, we get

$$c_k = [z^{-k}] \exp\left(\beta_1 z^{-1}\right) \cdots \exp\left(\beta_k z^{-k}\right) = \sum_{\substack{\ell_1,\ldots,\ell_k\geq 0 \\ \sum_{m=1}^k m\ell_m = k}} \prod_{m=1}^{k} \frac{\beta_m^{\ell_m}}{\ell_m!} \, .$$

We could make the expression still a little bit more efficient by using the fact that β_k vanishes for even $k \geq 1$.

(e) *Use graphs to compute the value of the second Bernoulli number B_2.*

Taking the log of a graph generating function passes from disconnected to connected graphs. So from Theorem 13, we know that

$$\log\left(\sqrt{\frac{z}{2\pi}}\int_{-\infty}^{\infty} e^{z(1+x-e^x)}\mathrm{d}x\right) \sim \sum_{k\geq 0} c_k^{\mathrm{cntd}} z^{-k},$$

where

$$c_k^{\mathrm{cntd}} = \sum_{\substack{G\in\mathcal{G}^u_{\mathrm{cntd}}|_{\deg\geq 3}\\ \chi(G)=-k}} \frac{(-1)^{|V_G|}}{|\operatorname{Aut}(G)|}\,.$$

We also know from the discussion above that

$$\log\left(\sqrt{\frac{z}{2\pi}}\int_{-\infty}^{\infty} e^{z(1+x-e^x)}\mathrm{d}x\right) = \log\left(\frac{\Gamma(z)}{\sqrt{\frac{2\pi}{z}}e^{-z}z^z}\right) \sim \sum_{k\geq 1}\beta_k z^{-k}\,.$$

It follows that $c_k^{\mathrm{cntd}} = \beta_k$. For $k=1$, we find via the same set of graphs from exercise 1.c) that

$$c_1^{\mathrm{cntd}} = c_1 = \frac{1}{12} = \frac{B_2}{1\cdot 2} \Rightarrow B_2 = \frac{1}{6}\,.$$

(f) *The Bernoulli numbers B_k vanish if k is odd and larger than 1. Use this to prove a vanishing statement for an alternating sum over connected graphs of fixed Euler characteristic.*

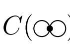
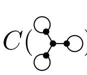
It follows that β_k vanishes for even k larger than 1. So, $c_k^{\mathrm{cntd}} = 0$ for all even k. (The empty graph does not count as a connected graph, hence $c_0^{\mathrm{cntd}} = 0$.) In graph sums, we have for all even k:

$$0 = \sum_{\substack{G\in\mathcal{G}^u_{\mathrm{cntd}}|_{\deg\geq 3}\\ \chi(G)=-k}} \frac{(-1)^{|V_G|}}{|\operatorname{Aut}(G)|}\,.$$

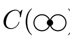
2. *Let $p_1, p_2, \ldots$ be an infinite set of variables. For a permutation $\alpha\in\operatorname{Sym}(H)$ with c_1^α 1-cycles, c_2^α 2-cycles, etc, we define the monomial $p^\alpha = \prod_{k\geq 0} p_k^{c_k^\alpha}$. For a given H-labeled graph G, the polynomial*

$$C(G) = \frac{1}{|\operatorname{Aut}(G)|}\sum_{\alpha\in\operatorname{Aut}(G)} p^\alpha,$$

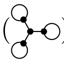
is the character *of the representation of* $\operatorname{Sym}(H)$ *associated with G. It is also known as the* Pólya cycle-index polynomial.

(a) *Compute $C($$)$ and $C($$)$.*

$\operatorname{Aut}($$)$ is a group of cardinality 8. We have two generators that flip each of the paddles (self-loops) on either side. They will contribute a factor of $\frac{1}{2}(\frac{1}{2}(p_1^2+p_2))^2 = \frac{1}{8}p_1^4+\frac{1}{8}p_2^2+\frac{1}{4}p_1^2p_2$ to the character. Here, the term $\frac{1}{2}p_1^2$ corresponds to fixing one paddle, and $\frac{1}{2}p_2$ corresponds to flipping a paddle. Because both generators commute, we can combine the characters by multiplication. Additionally, we have the automorphisms that switch the two paddles. These contribute the term $\frac{1}{4}(p_2^2+p_4)$ to the character, where the first term comes from switching the paddles such that the half-edges are permuted in two two-cycles, and the second term comes from switching the paddles such that the half-edges permute in one four-cycle. Hence,

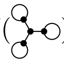
$C($$) = \frac{1}{8}(p_1^4 + 2p_1^2p_2 + 3p_2^2 + 2p_4).$

Analogously, we find



$$C(\quad) = \frac{1}{3!}\left(\frac{1}{2}p_1^2(p_1^2+p_2)\right)^3 + \frac{1}{2}\cdot\frac{1}{2}p_1^2(p_1^2+p_2)\cdot\frac{1}{2}p_2^2(p_2^2+p_4) + \frac{1}{3}\cdot\frac{1}{2}p_3^2(p_3^2+p_6),$$

where the first term covers all automorphisms where all non-selfloop edges are fixed, the second term covers those automorphisms that fix exactly one non-selfloop edge, and the last term covers the automorphisms that cyclically permute all non-selfloop edges. By expanding, we get



$$C(\quad) = \frac{1}{48}\left(p_1^{12}+3p_1^{10}p_2+3p_1^8p_2^2+p_1^6p_2^3+6p_1^4p_2^4+6p_1^4p_2^2p_4+6p_1^2p_2^5+6p_1^2p_2^3p_4+8p_3^4+8p_3^2p_6\right)$$

A cross-check is that $C(G)=1$ if we set $p_k=1$.

(b) *Observe that if $p_k=1$ for all $k\geq 1$, then $C(G)$ is 1. Prove that $C(G)\in\mathbb{Z}$ if $p_k=-1$ for all $k\geq 1$.*
We need to show that

$$\frac{1}{|\operatorname{Aut}(G)|}\sum_{\alpha\in\operatorname{Aut}(G)}(-1)^{\sum_k c_k^\alpha}\in\mathbb{Z}\,.$$

Recall that the sign of a permutation equals (-1) to the power of the number of even cycles of a permutation. Our graphs have an even number of half-edges. So, α always has an even number of odd cycles. Therefore, $\operatorname{sign}(\alpha)=(-1)^{\sum_k c_k^\alpha}$. If there is no automorphism $\beta\in\operatorname{Aut}(G)$ with $\operatorname{sign}(\beta)=-1$, then

$$\frac{1}{|\operatorname{Aut}(G)|}\sum_{\alpha\in\operatorname{Aut}(G)}\operatorname{sign}(\alpha)=1\in\mathbb{Z}\,.$$

So it only remains to prove the statement if there is such a $\beta\in\operatorname{Aut}(G)$ for which $\operatorname{sign}(\beta)=-1$, which we will assume from now on. Consider the map $\operatorname{sign}:\operatorname{Aut}(G)\to\{\pm1\}$, which is a surjective group homomorphism. As such, we might know that the preimages $\operatorname{sign}^{-1}(1)$ and $\operatorname{sign}^{-1}(-1)$ have the same cardinality by standard group theory. A more combinatorial way of seeing this is the following: Fix some $\beta\in\operatorname{Aut}(G)$ with $\operatorname{sign}(\beta)=-1$. The function $f:\alpha\mapsto\beta\alpha$ maps elements in $\operatorname{sign}^{-1}(1)$ to $\operatorname{sign}^{-1}(-1)$. This function has the inverse $f^{-1}:\alpha\mapsto\beta^{-1}\alpha$, so we constructed a bijection between $\operatorname{sign}^{-1}(1)$ and $\operatorname{sign}^{-1}(-1)$. Hence,

$$\sum_{\alpha\in\operatorname{Aut}(G)}\operatorname{sign}(\alpha)=|\operatorname{sign}^{-1}(1)|-|\operatorname{sign}^{-1}(-1)|=0\in\mathbb{Z}\,.$$

3. *Let $f(x)$ and $g(x)$ be power series $f(x)=1+\sum_{n\geq1}f_nx^n$, and $g(x)=\sum_{n\geq1}g_nx^n$ in $\mathbb{Q}[[x]]$, related by $f(x)=\exp(g(x))$.*

(a) *Prove a recursion that computes f_n when g_n is known. Also, prove a recursion that computes g_n when f_n is known.*
From $f(x)=\exp(g(x))$ follows $f'(x)=\exp(g(x))\cdot g'(x)=f(x)\cdot g'(x)$. Expanding both sides as power series gives,

$$\sum_{n\geq1}nf_nx^{n-1}=\sum_{m,k\geq0}f_mkg_kx^{m+k-1}\,,$$

where we agree that $f_0=1$. Therefore,

$$nf_n=\sum_{k=1}^{n}kf_{n-k}g_k=ng_n+\sum_{k=1}^{n-1}kf_{n-k}g_k$$

If all g_n are known, then we can recursively compute f_n via

$$f_n=g_n+\frac{1}{n}\sum_{k=1}^{n-1}kf_{n-k}g_k\quad\text{for all } n\geq1\,. \tag{3}$$

If all f_n are known, then we can recursively compute g_n via

$$g_n = f_n - \frac{1}{n}\sum_{k=1}^{n-1} k f_{n-k} g_k \quad \text{for all } n \geq 1\,. \tag{4}$$

(b) *Prove: If $g_n > 0$, then $f_n > 0$. Also prove: If $f_n < 0$, then $g_n < 0$.*

If $g_n > 0$, $f_n > 0$ follows recursively from eq. (3), because all terms on the right-hand side have a positive sign.

If $f_n < 0$, $g_n < 0$ follows recursively from eq. (4), because in total all terms on the right-hand side have a negative sign.

(c) *Find a graph-based expression for the $z \to \infty$ asymptotic expansion of*

$$I(z) = \sqrt{\frac{z}{2\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp\left(-z\frac{\sin^2(x)}{2}\right) \mathrm{d}x\,. \tag{5}$$

Identities of sin and cos give $-\frac{1}{2}\sin^2(x) = \frac{1}{4}(\cos(2x)-1) = \frac{1}{4}\sum_{k\geq 1}(-1)^k\frac{(2x)^{2k}}{(2k)!} = -\frac{x^2}{2} + \sum_{k\geq 3}\lambda_k\frac{x^k}{k!}$, where $\lambda_{2k} = (-1)^k 4^{k-1}$ and $\lambda_{2k+1} = 0$. Elementary trigonometry also tells us that $x = 0$ is the unique supremum of $-\frac{1}{2}\sin^2(x) = \frac{1}{4}(\cos(2x)-1)$. The integral $I(z)$ is finite for all $z > 0$, because $-\sin^2(x) \leq 0$ and

$$I(z) \leq \sqrt{\frac{z}{2\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp\left(z \cdot 0\right) \mathrm{d}x = \sqrt{\frac{z\pi}{2}}\,.$$

Hence, we may apply Theorem 13 to get the sum-over-graphs formula for the large z asymptotic expansion $I(z) \sim \sum_{k\geq 0} c_k z^{-k}$,

$$c_k = \sum_{\substack{G\in\mathcal{G}^u|_{\deg\geq 4}\\ \chi(G)=-k}} \frac{\prod_{v\in V_G}\lambda_{|v|}}{|\operatorname{Aut}(G)|}\,,$$

where λ_k is defined as above. We may also restrict the sum to only even graphs.

(d) *Compute the asymptotic expansion combinatorially, neglecting all terms in $\mathcal{O}(z^{-2})$.*

Recall 1.c). For $k = 0$, we only need the empty graph and $c_0 = 1$. For $k = 1$, we only need to compute the contribution of the graph, as the other relevant graphs have vertices of odd degree. That graph has only a four-valent vertex and $\lambda_4 = 4$. Therefore, $c_2 = \frac{4}{|\operatorname{Aut}(\cdot)|} = \frac{1}{2}$. In total, $I(z) = 1 + \frac{1}{2}z^{-1} + \mathcal{O}(z^{-2})$ for $z \to \infty$.

(e) *Prove a closed-form expression for the coefficients of this asymptotic expansion.*

We make the integral substitution $y = \sin(x)$ in (5). Recall that $\sin^2(x) + \cos^2(x) = 1$ and therefore $\cos(x) = \sqrt{1-\sin^2(x)}$ for $x \in [-\pi/2, \pi/2]$. We have $\mathrm{d}y = \cos(x)\mathrm{d}x \Rightarrow \mathrm{d}x = \frac{\mathrm{d}y}{\sqrt{1-y^2}}$. So, get from (5):

$$I(z) = \sqrt{\frac{z}{2\pi}} \int_{-1}^{1} e^{-z\frac{y^2}{2}} \frac{\mathrm{d}y}{\sqrt{1-y^2}}\,.$$

We know from Exercise 3.1 that

$$\int_{\varepsilon(z)}^{\infty} e^{-zy^2/2}\mathrm{d}y \in \mathcal{O}(z^{-R}) \text{ for all } R \geq 0.$$

if $\varepsilon(z) = z^{-\frac{5}{12}}$. Because $1/\sqrt{1-y^2} \leq 1$ in the integration domain, it follows that for all $R \geq 0$,

$$I(z) = \sqrt{\frac{z}{2\pi}} \int_{-\varepsilon(z)}^{\varepsilon(z)} e^{-z\frac{y^2}{2}} \frac{\mathrm{d}y}{\sqrt{1-y^2}} + \mathcal{O}(z^{-R})\,.$$

We may expand the algebraic part of the integrand via the generalized binomial theorem,

$$\frac{1}{\sqrt{1-y^2}} = \sum_{k=0}^{M-1} \binom{-\frac{1}{2}}{k} (-y)^{2k} + \mathcal{R}_M(y)$$

On the sufficiently small interval $y \in [-\varepsilon(z), \varepsilon(z)]$, the remainder term is bounded by $|\mathcal{R}_M(y)| \leq C\varepsilon(z)^{2M}$, where C is some constant independent of z. We get

$$I(z) = \sum_{k=0}^{M-1} \sqrt{\frac{z}{2\pi}} \int_{-\varepsilon(z)}^{\varepsilon(z)} e^{-z\frac{y^2}{2}} \binom{-\frac{1}{2}}{k} (-y)^{2k} \mathrm{d}y + \mathcal{O}(\varepsilon(z)^M) + \mathcal{O}(z^{-R}) .$$

We fix the expansion order M to be the smallest integer that is not smaller than $\frac{12}{10}R$. I.e. $M = \lceil \frac{12}{10}R \rceil$ and $\mathcal{O}(\varepsilon(z)^M) \subset \mathcal{O}(z^{-R})$. Again, from Exercise 3.1, we know that extending the integration domains of the Gaussian integral above to $\mathbb{R}$ only changes the expression by negligible $\mathcal{O}(z^{-R})$-terms. So,

$$\begin{aligned} I(z) &= \sum_{k=0}^{M-1} \sqrt{\frac{z}{2\pi}} \int_{-\infty}^{\infty} e^{-z\frac{y^2}{2}} \binom{-\frac{1}{2}}{k} (-y)^{2k} \mathrm{d}y + \mathcal{O}(z^{-R}) . \\ &= \sum_{k=0}^{M-1} z^{-k} (-1)^k (2k-1)!! \binom{-\frac{1}{2}}{k} + \mathcal{O}(z^{-R}) . \end{aligned}$$

Now, we can truncate the sum at $R-1 \leq M-1$, as higher order terms are absorbed by the $\mathcal{O}(z^{-R})$. The binomial can be rewritten as follows:

$$\binom{-\frac{1}{2}}{k} = \frac{(-\frac{1}{2}) \cdot (-\frac{3}{2}) \cdots (-\frac{1}{2} - k + 1)}{1 \cdot 2 \cdots k} = (-2)^{-k} \frac{(1) \cdot (3) \cdots (2k-1)}{1 \cdot 2 \cdots k}$$

So,

$$c_k = \frac{((2k-1)!!)^2}{2^k k!} = \frac{((2k)!)^2}{2^{3k}(k!)^3} .$$

In passing, we also proved a nontrivial positivity result. From the sum over graphs formula, it is not clear that the c_k are always positive; however, we have proved that $c_k > 0$ for all $k \geq 0$.

4. *For a given H-labeled graph $G = (V, E)$, define $\mathbb{Q}$-vector spaces $\mathbb{Q}H, \mathbb{Q}V, \mathbb{Q}E$ that are generated by the respective sets. Consider the linear map $\partial : \mathbb{Q}H \to \mathbb{Q}V \oplus \mathbb{Q}E, h \mapsto v_h - e_h$ that maps a half-edge generator to the difference of the generators for the vertex and edge v_h, e_h to which h belongs.*

   (a) *Prove that* $\dim\ker\partial = \#C(G) - \chi(G)$, *where $\#C(G)$ is the number of connected components of G and $\chi(G) = |V_G| - |E_G|$ is the Euler characteristic. (Hint: Give a combinatorial interpretation to the cokernel of ∂.)*

   The cokernel of ∂ is the quotient vector space $\mathbb{Q}V \oplus \mathbb{Q}E/\mathrm{im}(\partial)$. An element of this quotient is an equivalence class $[a] \in \{b \in \mathbb{Q}V \oplus \mathbb{Q}E : b - a \in \mathrm{im}(\partial)\}$. So, one edge is in the same equivalence class as a vertex if both are connected by a half-edge. The vertex, in turn, is again in the same equivalence class as all edges that are incident to it, and so on. Hence, the equivalence classes in $\mathrm{im}(\partial)$ correspond to connected components of G.

   Here is an alternative proof of the statement of the last paragraph: We claim that the dual quotient space $(\mathbb{Q}V \oplus \mathbb{Q}E/\mathrm{im}(\partial))^*$ is spanned by linear functions $\mathbb{Q}V \oplus \mathbb{Q}E \to \mathbb{Q}$ that are piece-wise constant on connected components. Suppose there was some $f : \mathbb{Q}V \oplus \mathbb{Q}E \to \mathbb{Q}$ that is not constant on a connected component. Therefore, there must be some half-edge $h \in H$ such that $f(v) \neq f(e)$ for the vertex v and the edge e that are connected by h. It follows that $f(\partial h) \neq 0$, which implies that f is not a well-defined functional on $(\mathbb{Q}V \oplus \mathbb{Q}E/\mathrm{im}(\partial))^*$.

   The statement now follows from the formula $\dim\ker\partial - \dim\mathrm{coker}\partial = \dim\mathbb{Q}H - \dim\mathbb{Q}V \oplus \mathbb{Q}E = 2|E| - |V| - |E| = -\chi(G)$.

(b) *Let $G = (V, E)$ be an H-labeled tree. For a given automorphism $\alpha \in \mathrm{Aut}(G)$, let α_V and α_E be the permutations that α induces on the sets V and E. Show that* $\mathrm{sign}(\alpha) = \mathrm{sign}(\alpha_E)\,\mathrm{sign}(\alpha_V)$.

A tree has one connected component and Euler characteristic 1. We learn from the last statement that $\dim\ker\partial = 0$ and that therefore ∂ is injective. Further, we know that $\dim \mathbb{Q}H = \dim \mathbb{Q}V \oplus \mathbb{Q}E + 1$, because the cokernel of ∂ is one-dimensional.

We will look at two different bases for the vector space $\mathbb{Q}V \oplus \mathbb{Q}E$ and the matrix representations of an automorphism $\alpha \in \mathrm{Aut}(G)$ in these bases. The determinant is independent of the choice of basis, so both matrix representations must have the same determinant. The first basis is the canonical one $v_1, \dots, v_{|V|}, e_1, \dots, e_{|E|}$. The representative matrix $\alpha_{\mathbb{Q}V \oplus \mathbb{Q}E}$ is a permutation matrix that does not mix vertices and edges. Therefore, $\det \alpha_{\mathbb{Q}V \oplus \mathbb{Q}E} = \mathrm{sign}\,\alpha_V\,\mathrm{sign}\,\alpha_E$.

Now, we represent the same automorphism in a different basis. The vectors $\partial h_1, \dots, \partial h_n$ are all linearly independent in $\mathbb{Q}V \oplus \mathbb{Q}E$ because ∂ is injective. There are $|H| = |E| + |V| - 1$ of them, so we miss one independent vector to make a basis. The vector $c = \sum_e e + \sum_v v$ is linearly independent of all vectors ∂h_i, which follows from the same argument as the one in the last exercise. The automorphism α permutes the vectors ∂h_i, but leaves the vector c invariant. So, the associated matrix representation $\beta_{\mathbb{Q}V \oplus \mathbb{Q}E}$ has determinant $\mathrm{sign}(\alpha)$.

Alternatively we may use the short exact sequence $0 \to \mathbb{Q}H \xrightarrow{\partial} \mathbb{Q}V \oplus \mathbb{Q}E \to \mathbb{Q} \to 0$ to prove this statement.