CQFT A2

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1. In this question:

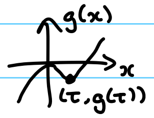
$$A_n = \sum_{\substack{G \in S^u | \deg \geq 3 \\ \chi(G) = -n}} \frac{\prod_{v \in V_G} \lambda_{|v|}}{|\mathrm{Aut}\, G|} \quad \text{for } \lambda_{|v|} = 1 \ \forall v \in G.$$

$\therefore$ since we would like to apply Theorem 15 to obtain an asymptotic expansion for A_n, we consider the entire fn:

$$g(x) = -\frac{x^2}{2} + \sum_{k \geq 3} \lambda_k \frac{x^k}{k!} = -\frac{x^2}{2} + \sum_{k \geq 3} \frac{x^k}{k!} = e^x - x^2 - x - 1$$

✓

$\therefore g(-x) \neq g(x)$, so we are in "case 1", i.e.



$$g(x+\tau) - g(\tau) = \sum_{k \geq 1} \frac{x^k}{k!} g^{(k)}(\tau) = (e^\tau - 2\tau - 1)x + (e^\tau - 2)\frac{x^2}{2} + \sum_{k \geq 3} e^\tau \frac{x^k}{k!}$$

$$= (e^\tau - 2)\frac{x^2}{2} + \sum_{k \geq 3} e^\tau \frac{x^k}{k!}$$ where τ is the unique positive solution to $2\tau = e^\tau - 1$, i.e. $g'(\tau) = 0$

$\therefore$ defining $\alpha = e^\tau - 2$, $\tilde{\lambda}_k = -e^\tau$ for $k \geq 3$:

$$g(\tau) - g(x+\tau) = -\alpha \frac{x^2}{2} + \sum_{k \geq 3} \tilde{\lambda}_k \frac{x^k}{k!}$$

$\therefore$ by Theorem 15, for $R = 2$:

$$A_n = w_0 (-\rho)^{-n} (n-1)! - w_1 (-\rho)^{-(n-1)} (n-2)! + \mathcal{O}((-\rho)^{-n} (n-3)!)$$

for $w_k = \frac{(-1)^k}{2\pi\sqrt{e^\tau - 2}} \sum_{\substack{G \in S^u | \deg \geq 3, \\ \chi(G) = -k}} (-1)^{|V_G|} \cdot \frac{e^{\tau |V_G|}}{|\mathrm{Aut}\, G|} (e^\tau - 2)^{-|E_G|}$, $k = 0, 1$, and $\rho := g(\tau) = (e^\tau - 2\tau - 1) - \tau^2 + \tau = -\tau(\tau - 1)$

To find w_0:

$\chi(G) = |V_G| - |E_G| = 0$, $\#$half-edges $= 2|E_G| = \sum_{v \in V_G} |v| \geq 3|V_G|$

$\Rightarrow |E_G| \leq 0 \ \Rightarrow |E_G| = 0$, which is the <u>empty graph</u>

$$\therefore w_0 = \frac{1}{2\pi\sqrt{e^\tau - 2}} = 0.129396\ldots = 0.1294 \text{ (4.s.f)}$$

To find w_1:

$\chi(G) = |V_G| - |E_G| = -1$, $2|E_G| \geq 3|V_G| \Rightarrow |V_G| \leq 2$

$\therefore$ the possible graphs with $\deg G \geq 3$, $\chi(G) = -1$ are:

case 1 (): $|V_G| = 1$, $|E_G| = 2$, $|\mathrm{Aut}\, G| = 2!^2 \times 2 = 8$

case 2 (): $|V_G| = 2$, $|E_G| = 3$, $|\mathrm{Aut}\, G| = 3! \times 2 = 12$

case 3 (): $|V_G| = 2$, $|E_G| = 3$, $|\mathrm{Aut}\, G| = 2 \times 2 \times 2 = 8$

$$\therefore w_1 = \frac{-1}{2\pi\sqrt{e^\tau - 2}} \left(-\frac{e^\tau}{8(e^\tau - 2)^2} + \left(\frac{1}{12} + \frac{1}{8}\right)\frac{e^{2\tau}}{(e^\tau - 2)^3}\right) = \frac{1}{2\pi\sqrt{e^\tau - 2}} \left(-\frac{e^\tau}{8(e^\tau - 2)^2} + \frac{5}{24} \cdot \frac{e^{2\tau}}{(e^\tau - 2)^3}\right) = +0.0128161\ldots$$
$$= +0.0128 \text{ (4.s.f)}$$

$$\therefore A_n = \frac{1}{2\pi\sqrt{e^\tau - 2}} \left[(\tau(\tau - 1))^{-n} (n-1)! - (\tau(\tau - 1))^{-n+1} (n-2)! \left(-\frac{e^\tau}{8(e^\tau - 2)^2} + \frac{5}{24} \cdot \frac{e^{2\tau}}{(e^\tau - 2)^3}\right)\right]$$

$$= 0.129396\ldots \left[0.322186\ldots^{-n} (n-1)! + 0.099045\ldots \cdot 0.322186^{-n+1} (n-2)!\right] + \mathcal{O}(-0.322186\ldots^{-n} (n-3)!)$$

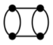
2. **8 marks**

Jerónimo Valencia

(a) For $\sigma_v, \sigma_w \in \{-1, +1\}$, prove the formula

$$\exp(\beta J \sigma_v \sigma_w) = \rho(1 + \kappa \sigma_v \sigma_w),$$

where $\rho = \cosh(\beta J)$ and $\kappa = \tanh(\beta J)$.

(b) Compute the partition function of the Ising model for the  and the  graph. For one of the two graphs, do the computation both by summing over all vertex configurations and by summing over all even subgraphs.

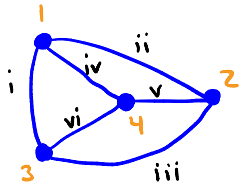
(a) Recall that $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$ and $\tanh(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$. Then

$$\rho(1 + K\sigma_v\sigma_w) = \frac{e^{\beta J} + e^{-\beta J}}{2}\left(1 + \frac{e^{\beta J} - e^{-\beta J}}{e^{\beta J} + e^{-\beta J}}\sigma_v\sigma_w\right)$$

$$= \frac{e^{\beta J} + e^{-\beta J}}{2} + \frac{e^{\beta J} - e^{-\beta J}}{2}\sigma_v\sigma_w = \frac{1 + \sigma_v\sigma_w}{2}e^{\beta J} + \frac{1 - \sigma_v\sigma_w}{2}e^{-\beta J}$$

Now we consider cases:

- If $\sigma_v = \sigma_w = \pm 1$ then $e^{\beta J\sigma_v\sigma_w} = e^{\beta J} = \cancelto{1}{\dfrac{1 + \sigma_v\sigma_w}{2}} e^{\beta J} + \cancelto{0}{\dfrac{1 - \sigma_v\sigma_w}{2}} e^{-\beta J}$
- If $\sigma_v = -\sigma_w = \pm 1$ then $e^{\beta J\sigma_v\sigma_w} = e^{-\beta J} = \cancelto{0}{\dfrac{1 + \sigma_v\sigma_w}{2}} e^{\beta J} + \cancelto{1}{\dfrac{1 - \sigma_v\sigma_w}{2}} e^{-\beta J}$

(b) $G =$ 

- Summing over vertex configurations:

$$Z(G, \beta J) = \underset{\hat{\sigma}:\ ++++}{\exp(\beta J \cdot (1+1+1+1+1+1))} + \underset{\hat{\sigma}:\ -+++}{\exp(\beta J \cdot (-1-1+1-1+1+1))}$$

$$+ \underset{\hat{\sigma}:\ +-++}{\exp(\beta J \cdot (1-1-1+1-1+1))} + \underset{\hat{\sigma}:\ ++-+}{\exp(\beta J \cdot (1+1-1-1+1-1))}$$

$$+ \underset{\hat{\sigma}:\ +++-}{\exp(\beta J \cdot (1+1+1-1-1-1))} + \underset{\hat{\sigma}:\ --++}{\exp(\beta J \cdot (-1+1-1-1-1+1))}$$

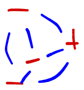
$$+ \underset{\hat{\sigma}:\ -+-+}{\exp(\beta J \cdot (1-1-1-1+1-1))} + \underset{\hat{\sigma}:\ -++-}{\exp(\beta J \cdot (-1-1+1+1-1-1))}$$

$$+ \underset{\hat{\sigma}:\ +--+}{\exp(\beta J \cdot (-1-1+1+1-1-1))} + \underset{\hat{\sigma}:\ +-+-}{\exp(\beta J \cdot (1-1-1-1+1-1))}$$

$$+ \underset{\hat{\sigma}:\ ++--}{\exp(\beta J \cdot (-1+1-1-1-1+1))} + \underset{\hat{\sigma}:\ ---+}{\exp(\beta J \cdot (1+1+1-1-1-1))}$$

$$+ \underset{\hat{\sigma}:\ --+-}{\exp(\beta J \cdot (-1+1-1+1+1-1))} + \underset{\hat{\sigma}:\ -+--}{\exp(\beta J \cdot (1-1-1+1-1+1))}$$

$$+ \underset{\hat{\sigma}:\ +---}{\exp(\beta J \cdot (-1-1+1-1+1+1))} + \underset{\hat{\sigma}:\ ----}{\exp(\beta J \cdot (1+1+1+1+1+1))}$$

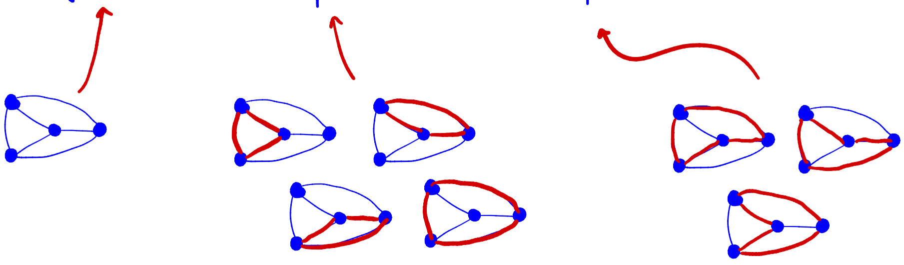


$$= 2\exp(6\beta J) + 6\exp(-2\beta J) + 8.$$

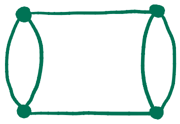
- Summing over all even subgraphs

$$Z_G = \cosh(\beta J)^6 \cdot 2^4 \sum_{\substack{\gamma \subseteq G \\ \gamma \text{ even}}} \tanh(\beta J)^{|E_\gamma|}$$

$$= \cosh(\beta J)^6 \cdot 2^4 \left(1 + 4 \tanh(\beta J)^3 + 3 \tanh(\beta J)^4 \right)$$



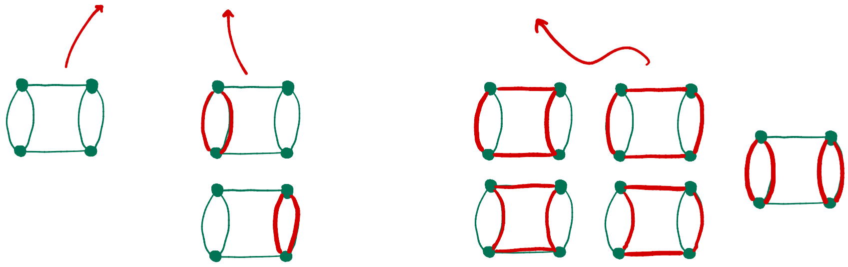
$G =$



- Summing over all even subgraphs

$$Z_G = \cosh(\beta J)^6 \cdot 2^4 \sum_{\substack{\gamma \subseteq G \\ \gamma \text{ even}}} \tanh(\beta J)^{|E_\gamma|}$$

$$= \cosh(\beta J)^6 \cdot 2^4 \left(1 + 2 \tanh(\beta J)^2 + 5 \tanh(\beta J)^4 \right)$$



$$= 8 + 2\exp(2\beta J) + 2\exp(-2\beta J) + 2\exp(6\beta J) + 2\exp(-6\beta J).$$

Problem 3.

Solution. We will first note that we may take out the α factor from the sum in question:

$$\begin{aligned}\sum_{k=R}^{n-R}\Gamma_\beta^\alpha(k)\Gamma_\beta^\alpha(n-k) &= \sum_{k=R}^{n-R}\alpha^{-k-\beta}\Gamma(k+\beta)\alpha^{-n+k-\beta}\Gamma(n-k+\beta)\\ &= \alpha^{-n-2\beta}\sum_{k=R}^{n-R}\Gamma(k+\beta)\Gamma(n-k+\beta).\end{aligned}$$

As $\Gamma_\beta^\alpha(n-R)=\alpha^{-n+R-\beta}\Gamma(n-R+\beta)$, the statement we are trying to prove follows from

$$\sum_{k=R}^{n-R}\Gamma(k+\beta)\Gamma(n-k+\beta)\in\mathcal{O}(\Gamma(n-R+\beta)), \tag{1}$$

since the factors of α^{-n} cancel out and the remaining factors involving α do not depend on n.

We now try to prove (1). Our first step is to prove the following inequality, which we claim holds by log-convexity of Γ:

$$\Gamma(k+\beta)\Gamma(n-k+\beta)\leq\Gamma(R+1+\beta)\Gamma(n-R-1+\beta),\quad k\in[R+1+\beta,n-R-1+\beta] \tag{2}$$

Consider the function $f(k)=\log\Gamma(k+\beta)+\log\Gamma(n-k+\beta)$ along the interval $[R+1+\beta,n-R-1+\beta]$. Since $\log\Gamma$ is convex, $f(k)$ is convex. Moreover, it is symmetric about the line $k=\frac{n}{2}$. Also, it is clear that $f(k)$ takes its minimum at $k=\frac{n}{2}$ and so $f(k)$ takes its maximums at the boundaries of the interval we are considering. This proves (2). Now observe that

$$\begin{aligned}\sum_{k=R}^{n-R}\Gamma(k+\beta)\Gamma(n-k+\beta) &= 2\Gamma(R+\beta)\Gamma(n-R+\beta)+\sum_{k=R+1}^{n-R-1}\Gamma(k+\beta)\Gamma(n-k+\beta)\\ &\leq 2\Gamma(R+\beta)\Gamma(n-R+\beta)+\sum_{k=R+1}^{n-R-1}\Gamma(R+1+\beta)\Gamma(n-R-1+\beta)\\ &= 2\Gamma(R+\beta)\Gamma(n-R+\beta)+(n-2R-1)\Gamma(R+1+\beta)\Gamma(n-R-1+\beta)\\ &= 2\Gamma(R+\beta)\Gamma(n-R+\beta)+(-R-\beta)\Gamma(R+1+\beta)\Gamma(n-R+\beta)\\ &= (2\Gamma(R+\beta)+(-R-\beta)\Gamma(R+1+\beta))\Gamma(n-R+\beta).\end{aligned}$$

Note that we used the fact that $z\Gamma(z)=\Gamma(z+1)$ on the second to last line. We have bounded the sum that appears in (1) by a constant multiple of $\Gamma(n-R+\beta)$, proving (1). ☐

Problem 4 *(14 marks)***.** The formula for the partition function of the Ising model of a graph G is

$$Z_G(\beta, J) = \sum_{\sigma \in \{-1,+1\}^{V_G}} \exp\left(\beta J \sum_{\substack{e \in E_G \\ e \cong (u,v)}} \sigma_u \sigma_v \right).$$

The partition function of the Ising model on *random 3-regular graphs* is the power series in z^{-1} that sums $Z_G(\beta, J)$ over all graphs:

$$Z(z, \beta, J) = \sum_G \frac{Z_G(\beta, J)}{|\mathrm{Aut}(G)|} z^{\chi(G)},$$

where the sum is over all 3-regular unlabeled, possibly disconnected graphs G without legs.

(a) Show that $Z(z, \beta, J)$ is a power series in $\mathbb{Q}[\kappa][[\rho^3 z^{-1}]]$ with $\rho = \cosh(\beta J)$ and $\kappa = \tanh(\beta J)$. (As always, you can refer to arguments from the lecture and do not have to replicate them.)

The even subgraph expansion of the Ising model partition function states that

$$Z_G(\beta, J) = \rho^{|E_G|} 2^{|V_G|} \sum_{H \text{ even subgraph of } G} \kappa^{|E_H|}. \tag{1}$$

In a 3-regular graph, we have $2|E_G| = 3|V_G|$, so $|V_G| = \frac{2}{3}|E_G|$. and $\chi(G) = |V_G| - |E_G| = -\frac{1}{3}|E_G|$. Writing everything in terms of $\chi(G)$, we have $|E_G| = -3\chi(G)$ and $|V_G| = -2\chi(G)$. Therefore, for a 3-regular graph G, we have

$$Z_G(\beta, J) = \rho^{-3\chi(G)} 2^{-2\chi(G)} \sum_{H \text{ even subgraph of } G} \kappa^{|E_H|}.$$

Therefore, the partition function of the Ising model on random 3-regular graphs is

$$\begin{aligned} Z(z, \beta, J) &= \sum_G \frac{Z_G(\beta, J)}{|\mathrm{Aut}(G)|} z^{\chi(G)} \\ &= \sum_G \frac{\rho^{-3\chi(G)} 2^{-2\chi(G)} \sum_{H \text{ even subgraph of } G} \kappa^{|E_H|}}{|\mathrm{Aut}(G)|} z^{\chi(G)} \\ &= \sum_G \frac{2^{-2\chi(G)} \sum_{H \text{ even subgraph of } G} \kappa^{|E_H|}}{|\mathrm{Aut}(G)|} (\rho^3 z^{-1})^{-\chi(G)} \end{aligned}$$

which is a power series in $\mathbb{Q}[\kappa][[\rho^3 z^{-1}]]$.

(b) Compute $Z(z, \beta, J)$ ignoring all terms in $\mathcal{O}(z^{-2})$ for $z \to \infty$.

If $Z(z, \beta, J) = c_0 + c_1 z^{-1} + \mathcal{O}(z^{-2})$, then

$$c_0 = \sum_{\substack{G \in \mathcal{G}^u|_{\text{3-regular}} \\ \chi(G)=0}} \frac{Z_G(\beta, J)}{|\mathrm{Aut}(G)|},$$

and

$$c_1 = \sum_{\substack{G \in \mathcal{G}^u|_{\text{3-regular}} \\ \chi(G)=-1}} \frac{Z_G(\beta, J)}{|\mathrm{Aut}(G)|}.$$

The only 3-regular graph with Euler characteristic 0 is the empty graph, which has $Z_G(\beta, J) = 1$ and $|\mathrm{Aut}(G)| = 1$. Therefore,

$$c_0 = 1.$$

For Euler characteristic -1, a 3-regular graph must have 2 vertices and 3 edges. There are two such graphs: $G_1 =$  and $G_2 =$ . Let v and w be the two vertices of G_1 and of G_2. We compute $|\mathrm{Aut}(G_1)| = 12$ and $|\mathrm{Aut}(G_2)| = 8$. The partition function of the Ising model on G_1 is

$$Z_{G_1}(\beta, J) = \sum_{\sigma \in \{-1,+1\}^{V_{G_1}}} \exp(\beta J 3 \sigma_v \sigma_w) = 2\exp(3\beta J) + 2\exp(-3\beta J), \tag{2}$$

and the partition function of the Ising model on G_2 is

$$Z_{G_2}(\beta, J) = \sum_{\sigma \in \{-1,+1\}^{V_{G_2}}} \exp(\beta J(\sigma_v \sigma_w + 2)) = 2\exp(3\beta J) + 2\exp(\beta J). \tag{3}$$

Therefore,

$$\begin{aligned} c_1 &= \frac{Z_{G_1}(\beta, J)}{|\mathrm{Aut}(G_1)|} + \frac{Z_{G_2}(\beta, J)}{|\mathrm{Aut}(G_2)|} \\ &= \frac{2\exp(3\beta J) + 2\exp(-3\beta J)}{12} + \frac{2\exp(3\beta J) + 2\exp(\beta J)}{8} \\ &= \underset{12}{\frac{5}{\cancel{24}}} \exp(3\beta J) + \frac{1}{4}\exp(\beta J) + \frac{1}{6}\exp(-3\beta J). \end{aligned}$$

Therefore,

$$Z(z, \beta, J) = 1 + \left(\underset{12}{\frac{5}{\cancel{24}}} \exp(3\beta J) + \frac{1}{4}\exp(\beta J) + \frac{1}{6}\exp(-3\beta J) \right) z^{-1} + \mathcal{O}(z^{-2}).$$

(c) Briefly explain why the following generating function identity holds:

$$\sum_G \frac{x^{n(G)}}{|\mathrm{Aut}(G)|} = \frac{1}{2} \log \frac{1}{1-x},$$

where the sum is over all connected, 3-regular, bridgeless unlabeled graphs of Euler characteristic 0 with arbitrarily many legs and $n(G)$ is the number of legs of G. Recall that the empty graph is not connected.

Let G be a connected, 3-regular, bridgeless unlabeled graph of Euler characteristic 0 with $n(G)$ legs. Since the Euler characteristic is 0, we have $|V_G| = |E_G|$. Since the graph is 3-regular, we have $3|V_G| = 2|E_G| + n(G)$. Therefore, $n(G) = |V_G| = |E_G|$.

This is impossible for $n = 0$, since the empty graph is not connected.

If a vertex v has three legs, then there are at least three legs, so there are at least three vertices, so there is a vertex $u \neq v$. But there is no path from v to u (since all three half-edges incident to v are legs), so the graph is not connected.

If v has two legs, let e be unique edge incident to v. Let u be the other endpoint of e, which is necessarily different from v. Since G is bridgeless, there is a path from v to u that does not use e. But this is impossible, since the only edge incident to v that is not a leg is e.

Therefore, every vertex has at most one leg. Since the number of legs equals the number of vertices, every vertex has exactly one leg, and consequently each vertex is incident to exactly two edges (counting multiplicities). Therefore, removing all legs from G yields a connected 2-regular graph. Connected 2-regular graphs are cycles, so removing all legs from G yields a cycle of length n. Therefore, there exists a unique graph G satisfying the conditions with n legs for every $n \geq 1$, which is obtained by attaching one leg to each vertex of the cycle of length n. The automorphism group of this graph has size $2n$ (since there are n rotations and n reflections of the cycle that can be extended to automorphisms of the graph).

Therefore,

$$\sum_G \frac{x^{n(G)}}{|\mathrm{Aut}(G)|} = \sum_{n=1}^{\infty} \frac{x^n}{2n} = \frac{1}{2} \log \frac{1}{1-x}.$$

(d) Use the subgraph-sum variant of Theorem 12 (Theorem 14) to prove that

$$Z(z, \beta, J) = \sum_{s \geq 0} (z^{-1}\rho^3)^s (2s-1)!![x^{2s}] \exp\left(2z\rho^{-3}\frac{x^3}{3!} + \frac{1}{2}\log\frac{1}{1-2\kappa x}\right).$$

The correct convention is to say that a graph with legs is even if all of its vertices are incident to an even number of edges (NOT counting legs, counting edges possibly with multiplicities).

For a 3-regular graph, being even is equivalent for all its vertices to be incident to 0 or 2 edges.

The only connected 3-regular graph with legs that contains a vertex incident to 0 edges is the graph with one vertex and three legs. Any other connected 3-regular graph with legs must have all its vertices incident to 2 edges (counting multiplicities), and therefore (since it's connected) its edges form a cycle. Therefore, the connected even 3-regular graphs are exactly the graph with one vertex and three legs, and the connected, 3-regular, bridgeless graphs with legs of Euler characteristic 0.

Removing the connectedness condition, a 3-connected graph with legs is even if and only if all its connected components are even, i.e., all of its connected components are either the graph with one vertex and three legs, or connected, 3-regular, bridgeless graphs with legs of Euler characteristic 0.

Let $\phi\colon \mathcal{G}^{u,\ell} \to \mathbb{Q}[\kappa][[\rho^3 z^{-1}]]$ be the multiplicative function defined on connected graphs with legs as $\phi(\succ\!\!\!\!\bullet\!\!-) = 2z\rho^{-3}$, $\phi(\gamma) = (2\kappa)^{|E_\gamma|}$ for all connected, 3-regular, bridgeless graphs with legs γ of Euler characteristic 0, and $\phi(\gamma) = 0$ otherwise.

The graphs γ with $\phi(\gamma) \neq 0$ are the disjoint unions of copies of the graph with one vertex and three legs, and connected, 3-regular, bridgeless graphs with legs of Euler characteristic 0, i.e., the even subgraphs with legs of 3-regular graphs.

Since the graph with one vertex and three legs has no edges and Euler characteristic 1, while the connected, 3-regular, bridgeless graphs with legs of Euler characteristic 0 have Euler characteristic 0, we have that for an even subgraph with legs γ of a 3-regular graph G,

$$\begin{aligned}\phi(\gamma) &= (2z\rho^{-3})^{\chi(\gamma)}(2\kappa)^{|E_\gamma|}\\ &= (2z\rho^{-3})^{|V_\gamma|-|E_\gamma|}(2\kappa)^{|E_\gamma|}.\end{aligned}$$

We compute that

$$\begin{aligned}\sum_{\substack{\gamma\in\mathcal{G}^{u,\ell}\\ \text{connected}}} \frac{\phi(\gamma)}{|\operatorname{Aut}(\gamma)|}x^{n(\gamma)} &= 2z\rho^{-3}\frac{x^3}{3!} + \sum_{\substack{\gamma\in\mathcal{G}^{u,\ell}\\ \text{connected}\\ \text{3-regular}\\ \text{bridgeless}}} \frac{(2\kappa)^{|E_\gamma|}}{|\operatorname{Aut}(\gamma)|}x^{n(\gamma)}\\ &= 2z\rho^{-3}\frac{x^3}{3!} + \sum_{\substack{\gamma\in\mathcal{G}^{u,\ell}\\ \text{connected}\\ \text{3-regular}\\ \text{bridgeless}}} \frac{1}{|\operatorname{Aut}(\gamma)|}(2\kappa x)^{n(\gamma)}\\ &= 2z\rho^{-3}\frac{x^3}{3!} + \frac{1}{2}\log\frac{1}{1-2\kappa x}.\end{aligned}$$

Apply Theorem 14 with $w = z^{-1}\rho^3$ to obtain

$$\begin{aligned}\sum_{G\in\mathcal{G}^u}\tfrac{1}{|\operatorname{Aut}(G)|}\sum_{\gamma\subset G}\phi(\gamma)(z^{-1}\rho^3)^{|E_G|-|E_\gamma|} &= \sum_{s\geq 0}(z^{-1}\rho^3)^s(2s-1)!![x^{2s}]\exp\left(\sum_{\substack{\gamma\in\mathcal{G}^{u,\ell}\\ \text{connected}}}\tfrac{\phi(\gamma)}{|\operatorname{Aut}(\gamma)|}x^{n(\gamma)}\right) \qquad (\star)\\ &= \sum_{s\geq 0}(z^{-1}\rho^3)^s(2s-1)!![x^{2s}]\exp\left(2z\rho^{-3}\tfrac{x^3}{3!} + \tfrac{1}{2}\log\tfrac{1}{1-2\kappa x}\right).\end{aligned}$$

Note that the right-hand side of $(\star)$ is the right-hand side of the desired identity.

We now prove that the left-hand side of $(\star)$ equals $Z(z,\beta,J)$. The left-hand side of $(\star)$ equals

$$\begin{aligned}&\sum_{G\in\mathcal{G}^u}\tfrac{1}{|\operatorname{Aut}(G)|}\sum_{\gamma\subset G}\phi(\gamma)(z^{-1}\rho^3)^{|E_G|-|E_\gamma|}\\ &= \sum_{G\in\,\mathcal{G}^u|_{\text{3-regular}}}\tfrac{1}{|\operatorname{Aut}(G)|}\sum_{\substack{\gamma\subset G\\ \text{even}}}(2z\rho^{-3})^{|V_\gamma|-|E_\gamma|}(2\kappa)^{|E_\gamma|}(z^{-1}\rho^3)^{|E_G|-|E_\gamma|}\\ &= \sum_{G\in\,\mathcal{G}^u|_{\text{3-regular}}}\tfrac{1}{|\operatorname{Aut}(G)|}\sum_{\substack{\gamma\subset G\\ \text{even}}}2^{|V_G|}(z\rho^{-3})^{|V_G|-|E_G|}\kappa^{|E_\gamma|}\\ &= \sum_{G\in\,\mathcal{G}^u|_{\text{3-regular}}}\tfrac{1}{|\operatorname{Aut}(G)|}\rho^{|E_G|}2^{|V_G|}\sum_{\substack{\gamma\subset G\\ \text{even}}}\kappa^{|E_\gamma|}z^{\chi(G)}\\ &= Z(z,\beta,J).\end{aligned}$$

Therefore, the left-hand side of $(\star)$ equals $Z(z,\beta,J)$, which completes the proof.