Bounds and estimates for Feynman-perturbative expansions

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Renormalized asymptotic enumeration of Feynman diagrams

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Physical motivation

- Often, the perturbation expansions turn out to have **vanishing** radius of convergence!

- Dyson’s argument: Let

\[ F(\alpha) = a_0 + a_1 \alpha + a_2 \alpha^2 + \ldots \]  

be a physical quantity in QED which is calculated as a formal power series in \( \alpha \).

- If \( F \) is analytic at \( \alpha = 0 \) we can analytically continue to negative \( \alpha \), resulting in a QFT where equal charges attract.

- The fictitious QFT will have no stable ground state.  
  \( \Rightarrow \) contradiction \( \Rightarrow F(\alpha) \) cannot be analytic at \( \alpha = 0 \).
First step: Number of diagrams

- The divergence of the perturbative expansion is believed to be caused by the proliferation of Feynman diagrams.
- Feynman diagrams can be counted rather easily using zero-dimensional field theory.
- The integral

\[
Z(\hbar) = \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar}\left(-\frac{x^2}{2} + F(x)\right)}
\]

is to be interpreted as a formal power series Cvitanović et al. [1978], Argyres et al. [2001], Hurst [1952], Molinari and Manini [2006].
- Possible ‘interactions’ are encoded in $F(x)$. 
Example

\[ Z_{\text{stir}}(\hbar) := \frac{\Gamma \left( \frac{1}{\hbar} \right)}{\sqrt{2\pi \hbar \left( \frac{1}{\hbar} \right)^{1/2}}} e^{-\frac{1}{\hbar}} = \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi \hbar}} e^{\frac{1}{\hbar} \left( -\frac{x^2}{2} - (e^x - 1 - x - \frac{x^2}{2}) \right)} \]

- **Combinatorial integral** representation of Stirling’s famous (asymptotic) expansion of the Gamma-function.

- Counts the (orbifold) Euler characteristic of the moduli space of (stable) open curves Kontsevich [1992],

\[ \log Z_{\text{stir}}(\hbar) = \sum_{g,n} \frac{\chi(\mathcal{M}_{g,n})}{n!} \hbar^{n+2g-2} \]

\[ n+2g-2 \geq 0 \]
Example

\[ Z^{\text{stir}}(\hbar) := \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi \hbar}} e^{\frac{1}{\hbar}(\frac{-x^2}{2} - (e^x - 1 - x - \frac{x^2}{2}))} \]

- Set \( F(x) = -(e^x - 1 - x - \frac{x^2}{2}) \). Combinatorial: All vertices are allowed and \( \lambda_k = -1 \).
- Diagrammatically:

\[
Z^{\text{stir}}(\hbar) = 1 + \frac{1}{8} + \frac{1}{12} + \frac{1}{8} + \ldots
= 1 + \hbar \left( \frac{1}{8}(-1)^2 + \frac{1}{12}(-1)^2 + \frac{1}{8}(-1) \right) + \ldots
= 1 + \frac{1}{12}

= 1 + \hbar \frac{1}{12} + \hbar^2 \frac{1}{288} - \hbar^3 \frac{139}{51840} - \hbar^4 \frac{571}{2488320} + \ldots
\]
■ Defines a map $\mathcal{F} : x^3 \mathbb{R}[[x]] \to \mathbb{R}[[\hbar]]$.

■ Suitable for studying **random graphs** Erdös and Rényi [1959].

■ Efficient calculation is possible using an interpretation as a hyperelliptic curve.
Interpretation as hyperelliptic curve

\[ Z(\hbar) = \sum_{n=0}^{\infty} (2n - 1)!! [y^{2n}] G'(y) \]

where \( G(y) \) is the (positive) solution of \( \frac{y^2}{2} = \frac{G(y)^2}{2} - F(G(y)) \).

- The implicit equation \( \frac{y^2}{2} = \frac{G(y)^2}{2} - F(G(y)) \) defines a **complex curve** in \( \mathbb{C}^2 \).
- The asymptotics of \( Z(\hbar) \) are governed by the asymptotics of the **convergent** power series \( G(y) \).
- Similar structures to **topological recursion** Eynard and Orantin [2007].
Figure: Plot of the elliptic curve $\frac{y^2}{2} = \frac{x^2}{2} - \frac{x^3}{3!}$, which can be associated to the perturbative expansion of zero-dimensional $\phi^3$-theory. The dominant singularity can be found at $(x, y) = \left(2, \frac{2}{\sqrt{3}}\right)$. 

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Renormalization can be used to restrict the number of diagrams.

Using BPHZ renormalization, the number of skeleton diagrams is obtained.

More sophisticated techniques can be used to restrict to more general classes of diagrams ⇒ Hopf algebra of Feynman diagrams Connes and Kreimer [1999].

Answers question by Freeman Dyson: Number of skeleton diagrams in quenched QED is

\[ e^{-2(2n-1)!!} \left( 1 - \frac{6}{2n+1} - \frac{4}{(2n-1)(2n+1)} - \frac{218}{3} \frac{1}{(2n-3)(2n-1)(2n+1)} + \cdots \right), \]

Hopf algebra techniques can be used to evaluate random graph models.
There are many ways to impose bounds on the value of Feynman integrals Bender and Wu [1969].

Interesting algebraic structure: The ‘Hepp-bound’.

Renormalization group invariant part of the amplitude is bounded Panzer [2016]:

\[
P(\Gamma) = \int \frac{d\Omega}{\Psi^2} \leq \sum_{\emptyset \subset \gamma_1 \subset \cdots \subset \gamma_{n-1} \subset \Gamma} \frac{1}{\omega_D(\gamma_1) \cdots \omega_D(\gamma_n)}
\]

- Sum over all flags, maximal chains of 1PI subdiagrams of $\Gamma$.
- $\omega_D$ assigns the superficial degree of divergence to the subgraph $\gamma_i$. 
These bounds can be summed over all diagrams.

The generating function for the sum fulfills a non-linear ODE for instance in $\phi^4$ MB [2017]:

\[
\left( \frac{1}{2} x \partial_x - 1 \right) F(x) = \frac{1}{2} \hbar \left( \partial_x^2 \log \frac{1}{1 - F(x)} \right.
\]

\[
- \left. \left[ \left( 1 + \frac{x^2}{2} \partial_\xi^2 \right) \partial_\xi^2 \log \frac{1}{1 - F(\xi)} \right]_{\xi=0} \right)
\]

Also carries interesting Hopf-algebraic structures.

Related to combinatorial constructions on graphs: Ear decompositions and Fulkerson conjecture.
Summary

- Renormalization together with the divergence of the perturbation expansion shows very interesting mathematical structures.
- Hopf algebra techniques enable us to extend the notion of renormalization to evaluate restricted random graph models.
- Similar structures can be used to describe bounds for diagrams, which can be summed easily.
- Hints that we may setup approximations for Feynman integrals that become more accurate the larger the diagram gets.


CA Hurst. The enumeration of graphs in the feynman-dyson technique. In *Proceedings of the Royal Society of London A*:

