The Euler characteristic of $\text{Out}(F_n)$

Michael Borinsky, Nikhef
February 7, Emmy-Noether-Seminar - Universität Erlangen

joint work with Karen Vogtmann
arXiv:1907.03543
Introduction I: Groups
Automorphisms of groups

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- An **automorphism** of $G$, $\rho \in \text{Aut}(G)$ is a bijection

\[ \rho : G \to G \]

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$$g \mapsto h^{-1}gh$$

for each $h \in G$. 
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- Outer automorphisms: $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$
Automorphisms of the free group

Consider the free group with $n$ generators

$$F_n = \langle a_1, \ldots, a_n \rangle$$

E.g. $a_1a_3^{-5}a_2 \in F_n$
Automorphisms of the free group

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\[ F_n = \langle a_1, \ldots, a_n \rangle \]

E.g. \( a_1 a_3^{-5} a_2 \in F_n \)

• The group \( \text{Out}(F_n) \) is our main object of interest.
Some properties of $\text{Out}(F_n)$

- Generated by

\[ a_1 \mapsto a_1 a_2 \quad a_2 \mapsto a_2 \quad a_3 \mapsto a_3 \quad \ldots \]

and

\[ a_1 \mapsto a_1^{-1} \quad a_2 \mapsto a_2 \quad a_3 \mapsto a_3 \quad \ldots \]

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and permutations of the letters.

- The fundamental group of a graph is always a free group,

$$\text{Out}(F_n) = \text{Out}(\pi_1(\Gamma))$$

for a connected graph $\Gamma$ with $n$ independent cycles.
Another example of an outer automorphism group: 
the mapping class group
• Another example of an outer automorphism group: the **mapping class group**

• The group of homeomorphisms of a closed, connected and orientable surface $S_g$ of genus $g$ up to isotopies

\[ \text{MCG}(S_g) := \text{Out}(\pi_1(S_g)) \]
Example: Mapping class group of the torus

\[ \text{MCG}(\mathbb{T}^2) = \text{Out}(\pi_1(\mathbb{T}^2)) \]

The group of homeomorphisms \( \mathbb{T}^2 \to \mathbb{T}^2 \) up to an isotopy:

\[ \Rightarrow \quad \text{generated by Dehn twists} \quad \Rightarrow \quad \text{SL}(2, \mathbb{Z}) \]
Introduction II: Spaces
How to study such groups? 

How to study groups such as $\text{MCG}(S)$ or $\text{Out}(F_n)$?
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How to study groups such as $\text{MCG}(S)$ or $\text{Out}(F_n)$?

**Main idea**

Realize $G$ as symmetries of some geometric object.

Due to Stallings, Thurston, Gromov, ... (1970–)
For the mapping class group: Teichmüller space

Let $S$ be a closed, connected and orientable surface.
For the mapping class group: Teichmüller space

Let $S$ be a closed, connected and orientable surface.

$\Rightarrow$ A point in Teichmüller space $T(S)$ is a pair, $(X, \mu)$

- A Riemann surface $X$.
- A marking: a homeomorphism $\mu : S \to X$. 

![Diagram of a closed, connected, and orientable surface $S$ being mapped to a Riemann surface $X$ via a homeomorphism $\mu$.]
For the mapping class group: Teichmüller space

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MCG($S$) acts on $T(S)$ by composing to the marking:

$$(X, \mu) \mapsto (X, \mu \circ g^{-1})$$

for some $g \in \text{MCG}(S)$. 

For $\text{Out}(F_n)$: Outer space

Idea: Mimic previous construction for $\text{Out}(F_n)$.

Culler, Vogtmann (1986)
For $\text{Out}(F_n)$: Outer space

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Let $R_n$ be the rose with $n$ petals.

$R_3 = \text{rose diagram}$
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Let $R_n$ be the rose with $n$ petals.

$R_3 = \begin{array}{c}
\end{array}$

A point in Outer space $O_n$ is a pair, $(G, \mu)$

- A connected graph $G$ with a length assigned to each edge.
- A marking: a homotopy $\mu : R_n \rightarrow G$. 
For $\text{Out}(F_n)$: Outer space

Idea: Mimic previous construction for $\text{Out}(F_n)$.

*Culler, Vogtmann (1986)*

Let $R_n$ be the rose with $n$ petals.

\[ R_3 = \begin{array}{c} \text{Diagram of a rose with 3 petals} \end{array} \]

$\Rightarrow$ A point in Outer space $\mathcal{O}_n$ is a pair, $(G, \mu)$
- A connected graph $G$ with a length assigned to each edge.
- A marking: a homotopy $\mu : R_n \to G$.

Out$(F_n)$ acts on $\mathcal{O}_n$ by composing to the marking:

$$(\Gamma, \mu) \mapsto (\Gamma, \mu \circ g^{-1}) \text{ for some } g \in \text{Out}(F_n) = \text{Out}(\pi_1(R_n))$$
Examples of applications of Outer space

- The group $\text{Out}(F_n)$
- Moduli spaces of punctured surfaces
- Tropical curves
- Invariants of symplectic manifolds
- Classical modular forms
- (Mathematical) physics
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- The group $\text{Out}(F_n)$
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  Scalar QFT $\sim$ Integrals over $O_n / \text{Out}(F_n)$
Examples of applications of Outer space

• The group Out($F_n$)
• Moduli spaces of punctured surfaces
• Tropical curves
• Invariants of symplectic manifolds
• Classical modular forms
• (Mathematical) physics :

\[
\text{Scalar QFT} \sim \text{Integrals over } \mathcal{O}_n / \text{Out}(F_n)
\]

analogous to

\[
2D \text{ Quantum gravity} \sim \text{Integral over } T(S) / \text{MCG}(S)
\]
• The quotient space $\mathcal{G}_n := \mathcal{O}_n / \text{Out}(F_n)$ is called the moduli space of graphs.
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Its cousin $\mathcal{M}_g = T(S_g) / \text{MCG}(S_g)$ is the moduli space of curves.
Moduli spaces

- The quotient space $G_n := O_n / \text{Out}(F_n)$ is called the moduli space of graphs.
- Its cousin $\mathcal{M}_g = T(S_g) / \text{MCG}(S_g)$ is the moduli space of curves.
- Both can be used to study the respective groups.
## Summary of the respective groups and spaces

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<tr>
<th>Acts freely and properly on</th>
<th>$\text{MCG}(S_g)$</th>
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Invariants
Algebraic invariants

- $H_\bullet(\text{Out}(F_n); \mathbb{Q}) \cong H_\bullet(\mathcal{O}_n / \text{Out}(F_n); \mathbb{Q}) = H_\bullet(G_n; \mathbb{Q})$, as $\mathcal{O}_n$ is contractible Culler, Vogtmann (1986).
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• $H_\bullet(\text{Out}(F_n); \mathbb{Q}) \cong H_\bullet(\mathcal{O}_n / \text{Out}(F_n); \mathbb{Q}) = H_\bullet(\mathcal{G}_n; \mathbb{Q})$, as $\mathcal{O}_n$ is contractible Culler, Vogtmann (1986).

⇒ Study $\text{Out}(F_n)$ using $\mathcal{G}_n$!
Algebraic invariants

- \( H_\bullet(\text{Out}(F_n); \mathbb{Q}) \cong H_\bullet(\mathcal{O}_n / \text{Out}(F_n); \mathbb{Q}) = H_\bullet(\mathcal{G}_n; \mathbb{Q}) \), as \( \mathcal{O}_n \) is contractible \textbf{Culler, Vogtmann (1986)}.  

\( \Rightarrow \) Study \( \text{Out}(F_n) \) using \( \mathcal{G}_n \)!

- One simple invariant: Euler characteristic
Further motivation to look at Euler characteristic of $\text{Out}(F_n)$

Consider the abelization map $F_n \rightarrow \mathbb{Z}^n$. 
Further motivation to look at Euler characteristic of $Out(F_n)$

Consider the abelianization map $F_n \rightarrow \mathbb{Z}^n$.
⇒ Induces a group homomorphism

$Out(F_n) \rightarrow Out(\mathbb{Z}^n)$
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Consider the abelization map $F_n \rightarrow \mathbb{Z}^n$.
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$$\text{Out}(F_n) \rightarrow \text{Out}(\mathbb{Z}^n)$$

$$= \text{GL}(n, \mathbb{Z})$$
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$$1 \rightarrow \mathcal{T}_n \rightarrow \text{Out}(F_n) \rightarrow \underbrace{\text{Out}(\mathbb{Z}^n)}_{=\text{GL}(n,\mathbb{Z})} \rightarrow 1$$
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- $\mathcal{T}_n$ the ‘non-abelian’ part of Out($F_n$) is interesting.
Further motivation to look at Euler characteristic of $\text{Out}(F_n)$

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$$1 \rightarrow T_n \rightarrow \text{Out}(F_n) \rightarrow \underbrace{\text{Out}(\mathbb{Z}^n)}_{=\text{GL}(n, \mathbb{Z})} \rightarrow 1$$

- $T_n$ the ‘non-abelian’ part of $\text{Out}(F_n)$ is interesting.
- By the short exact sequence above

$$\chi(\text{Out}(F_n)) = \chi(\text{GL}(n, \mathbb{Z})) \chi(T_n)$$
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$$\chi(\text{Out}(F_n)) = \chi(\text{GL}(n, \mathbb{Z})) \chi(\mathcal{T}_n)$$

$$= 0$$

⇒ $\mathcal{T}_n$ does not have finitely-generated homology if $\chi(\text{Out}(F_n)) \neq 0$. 

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Based on initial computations by Smillie-Vogtmann (1987) up to $n \leq 11$. Later strengthened by Zagier (1989) up to $n \leq 100$. 
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**Conjecture Smillie-Vogtmann (1987)**

\[ \chi(\text{Out}(F_n)) \neq 0 \text{ for all } n \geq 2 \]

and \( |\chi(\text{Out}(F_n))| \) grows exponentially for \( n \to \infty \).

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In topological terms, i.e. \( \dim(H_2(\mathcal{T}_n)) = \infty \), which implies that \( \mathcal{T}_n \) does not have finitely-generated homology.

**Theorem** Bestvina, Bux, Margalit (2007)

\( \mathcal{T}_n \) does not have finitely-generated homology.
Results: $\chi(\text{Out}(F_n)) \neq 0$
\begin{align*}
\chi(\text{Out}(F_n)) &< 0 \text{ for all } n \geq 2
\end{align*}
Theorem A MB-Vogtmann (2019)

\[ \chi(\text{Out}(F_n)) < 0 \text{ for all } n \geq 2 \]

\[ \chi(\text{Out}(F_n)) \sim -\frac{1}{\sqrt{2\pi}} \frac{\Gamma(n - 3/2)}{\log^2 n} \text{ as } n \to \infty. \]
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- Only one odd-dimensional class known Bartholdi (2016).
- Where does all this homology come from?
This Theorem A follows from an implicit expression for $\chi(\text{Out}(F_n))$:
This Theorem A follows from an implicit expression for \( \chi(\text{Out}(F_n)) \):

**Theorem B MB-Vogtmann (2019)**

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\sqrt{2\pi} e^{-N} N^N \sim \sum_{k \geq 0} a_k (-1)^k \Gamma(N + 1/2 - k) \text{ as } N \to \infty
\]

where

\[
\sum_{k \geq 0} a_k z^k = \exp \left( \sum_{n \geq 0} \chi(\text{Out}(F_{n+1})) z^n \right)
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- An analytic argument is needed to prove Theorem A from Theorem B.
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- An analytic argument is needed to prove Theorem A from Theorem B.
- In this talk: Focus on proof of Theorem B.
Analogy to the mapping class group
Harer-Zagier formula for $\chi(\text{MCG}(S_g))$

Similar result for the mapping class group/moduli space of curves:
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**Theorem** Harer-Zagier (1986)

$$\chi(\mathcal{M}_g) = \chi(\text{MCG}(S_g)) = \frac{B_{2g}}{4g(g - 1)} \quad g \geq 2$$
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- Alternative proof using topological field theory (TFT) by Penner (1988).
- Simplified proof by Kontsevich (1992) based on TFT's.

$\Rightarrow$ Kontsevich's proof served as a blueprint for $\chi(\text{Out}(F_n))$. 

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Sketch of Kontsevich’s TFT proof of the Harer-Zagier formula
Step 1 of Kontsevich’s proof

Generalize from $\mathcal{M}_g$ to $\mathcal{M}_{g,n}$, the moduli space of surfaces of genus $g$ and $n$ punctures.
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Generalize from $M_g$ to $M_{g,n}$, the moduli space of surfaces of genus $g$ and $n$ punctures.

We can ‘forget one puncture’:

$$ \text{MCG}(S_{g,n+1}) \rightarrow \text{MCG}(S_{g,n}) $$
Generalize from $\mathcal{M}_g$ to $\mathcal{M}_{g,n}$, the moduli space of surfaces of genus $g$ and $n$ punctures.

We can ‘forget one puncture’:

$$1 \to \pi_1(S_{g,n}) \to \text{MCG}(S_{g,n+1}) \to \text{MCG}(S_{g,n}) \to 1$$

$$\Rightarrow \chi(\text{MCG}(S_{g,n+1})) = \chi(\mathcal{M}_{g,n+1}) = \chi(\pi_1(S_{g,n})) \chi(\mathcal{M}_{g,n})$$
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$$\Rightarrow \chi(\text{MCG}(S_{g,n+1})) = \chi(\mathcal{M}_{g,n+1}) = \chi(\pi_1(S_{g,n})) \chi(\mathcal{M}_g)$$

$$= 2 - 2g - n$$
Step 2 of Kontsevich’s proof

- Use a combinatorial model for $\mathcal{M}_{g,n}$
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- Use a combinatorial model for $\mathcal{M}_{g,n}$

$\Rightarrow$ Ribbon graphs Penner (1986)

Every point in $\mathcal{M}_{g,n}$ can be associated with a ribbon graph $\Gamma$ such that

- $\Gamma$ has $n$ boundary components: $h_0(\partial \Gamma) = n$
- $\chi(\Gamma) = |V_\Gamma| - |E_\Gamma| = 2 - 2g - n$. 
Step 2 of Kontsevich’s proof

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$\Rightarrow$ Ribbon graphs Penner (1986)

Every point in $\mathcal{M}_{g,n}$ can be associated with a ribbon graph $\Gamma$ such that

- $\Gamma$ has $n$ boundary components: $h_0(\partial \Gamma) = n$
- $\chi(\Gamma) = |V_\Gamma| - |E_\Gamma| = 2 - 2g - n$.

$\Rightarrow$ $\Gamma$ can be interpreted as a surface of genus $g$ with $n$ punctures.
Step 2 of Kontsevich’s proof

- Use a combinatorial model for $\mathcal{M}_{g,n}$

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sum over representatives of cells of \( M_{g,n} \)

\( \sim \) ribbon graphs \( \Gamma \)

dimension of resp. strata

\[ = \text{l.v.1 mod 2} \]

stabilizer under action of \( MCG \)

\[ = 1 \Delta n + \Pi \]
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subject to:
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Used by Penner (1988) to calculate \( \chi(M_g) \) with Matrix models.
Step 4 of Kontsevich’s proof

Kontsevich’s simplification:

$$\sum_{g,n} \frac{\chi(M_{g,n})}{n!} z^{2-2g-n}$$
Step 4 of Kontsevich’s proof

Kontsevich’s simplification:

\[
\sum_{g,n} \frac{\chi(M_{g,n})}{n!} z^{2g-n} = \sum \sum \frac{(-1)^{|V_\Gamma|}}{|\text{Aut} \Gamma|} \frac{1}{n!} z^{\chi(\Gamma)}
\]

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Step 4 of Kontsevich’s proof

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\]

\[
= \sum_{\text{graphs } G} \frac{(-1)^{|V_{\Gamma}|}}{|\text{Aut } G|} z^{\chi(G)}
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Step 4 of Kontsevich’s proof

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\sum_{g,n} \frac{\chi(M_{g,n})}{n!} z^{2-2g-n} = \sum_{g,n} \sum_{\text{ribbon graphs } \Gamma} \frac{(-1)^{|V_\Gamma|}}{|\text{Aut } \Gamma|} \frac{1}{n!} z^{\chi(\Gamma)}
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= \sum_{\text{graphs } G} \frac{(-1)^{|V_G|}}{|\text{Aut } G|} z^{\chi(G)}
\]

This is the perturbative series of a simple TFT:

\[
= \log \left( \frac{1}{\sqrt{2\pi z}} \int_{\mathbb{R}} e^{z(1+x-e^x)} dx \right)
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Evaluation is classic (Stirling/Euler-Maclaurin formulas)

\[
= \sum_{k \geq 1} \frac{\zeta(-k)}{-k} z^{-k}
\]
Last step of Kontsevich’s proof

\[ \sum_{g, n \atop 2-2g-n=k} \frac{\chi(\mathcal{M}_{g, n})}{n!} = \frac{B_{k+1}}{k(k+1)} \]
Last step of Kontsevich’s proof

\[
\sum_{g,n \mid 2-2g-n=k} \frac{\chi(\mathcal{M}_{g,n})}{n!} = \frac{B_{k+1}}{k(k+1)}
\]

\(\Rightarrow\) recover Harer-Zagier formula using the identity

\[\chi(\mathcal{M}_{g,n+1}) = (2 - 2g - n)\chi(\mathcal{M}_{g,n})\]
Analogous proof strategy for $\chi(\text{Out}(F_n))$ using renormalized TFTs
Generalize from $\text{Out}(F_n)$ to $A_{n,s}$ and from $\mathcal{O}_n$ to $\mathcal{O}_{n,s}$, Outer space of graphs of rank $n$ and $s$ legs.

Contant, Kassabov, Vogtmann (2011)
Step 1

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Contant, Kassabov, Vogtmann (2011)

Forgetting a leg gives the short exact sequence of groups

$$1 \rightarrow F_n \rightarrow A_{n,s} \rightarrow A_{n,s-1} \rightarrow 1$$
Step 2

- Use a combinatorial model for $G_{n,s}$
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A point in $G_{n,s}$ can be associated with a pair of a graph $G$ and a forest $f \subseteq G$. 
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\[(G, f)\]
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- Sum over representatives of cells of \( \Theta_{n,s}/A_{n,s} \)
- \( \Rightarrow \) legged graphs \( G \)
- Dimension of resp. strata \( = |E| \)
- Stabilizer under action of \( A_{n,s} \)
  \( = |A| + |G| \)
$$\chi(A_{n,s}) = \sum_{\sigma} \frac{(-1)^{\dim(\sigma)}}{|\text{Stab}(\sigma)|}$$

$$= \sum_{\text{graphs } G \text{ with } s \text{ legs}} \sum_{\text{forests } f \subset G} \frac{(-1)^{|E_f|}}{|\text{Aut } G|}$$

$$\text{rank}(\pi_1(G)) = n$$
Renormalized TFT interpretation MB-Vogtmann (2019):

\[ \chi(A_{n,s}) = \sum_{\text{graphs } G \text{ with } s \text{ legs}} \frac{1}{|\text{Aut } G|} \sum_{\text{forests } f \subset G} (-1)^{|E_f|} \]
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Step 4

Renormalized TFT interpretation MB-Vogtmann (2019):

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\( \tau \) fulfills the identities \( \tau(\emptyset) = 1 \) and

\[ \sum_{g \subset G, \text{ bridgeless}} \tau(g)(-1)^{|E_{G/g}|} = 0 \text{ for all } G \neq \emptyset \]
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The group invariants \(\chi(A_{n,s})\) are encoded in a renormalized TFT.
Let

\[ T(z, x) = \sum_{n, s \geq 0} \chi(A_{n,s}) z^{1-n} \frac{x^s}{s!} \]
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then \[ 1 = \frac{1}{\sqrt{2\pi z}} \int_{\mathbb{R}} e^{T(z,x)} \, dx \]
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Using the short exact sequence, \( 1 \to F_n \to A_{n,s} \to A_{n,s-1} \to 1 \) results in the action

\[
1 = \frac{1}{\sqrt{2\pi z}} \int_{\mathbb{R}} e^{z(1+x-e^x)+\frac{x}{2}+T(-xe^x)} \, dx
\]

where \( T(z) = \sum_{n \geq 1} \chi(\text{Out}(F_{n+1})) z^{-n} \).
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This gives the implicit result in Theorem B.
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Open questions:

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Summery/Questions/Outlook

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- Can the TFT analysis be explained with a duality between $\text{MCG}(S_g)$ and $\text{Out}(F_n)$? Obvious candidate: Koszul duality
- Can renormalized TFT arguments also be used for other groups? For instance RAAGs.