Hopf algebras and factorial divergent power series: Algebraic tools for graphical enumeration

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Overview

Motivation

- Perturbation expansions in quantum field theory may be organized as sums of integrals.
- Each integral maybe represented as a graph.
- Usually these expansions have vanishing radius of convergence. The coefficients behave as $f_n \approx C A^n \Gamma(n+\beta)$ for large n.
- Divergence is believed to be caused by proliferation of graphs.

- In zero dimensions the integration becomes trivial.
- Observables are generating functions of certain classes of graphs [Hurst, 1952, Cvitanović et al., 1978, Argyres et al., 2001, Molinari and Manini, 2006].
- For instance, the partition function in φ^3 theory is formally,

$$Z^{\varphi^3}(\hbar) = \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar}\left(-\frac{x^2}{2}+\frac{x^3}{3!}\right)}.$$

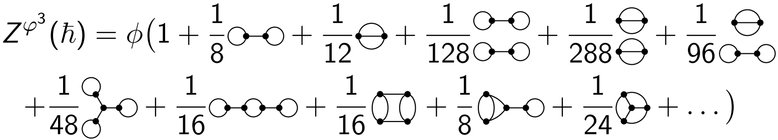
- 'Formal' $\Rightarrow$ expand under the integral sign and integrate over Gaussian term by term,

$$Z^{\varphi^3}(\hbar) := \sum_{n=0}^{\infty} \hbar^n (2n-1)!![x^{2n}] e^{\frac{x^3}{3!\hbar}}$$

- Z^{φ^3} is the exponential generating function of 3-valent multigraphs,

$$Z^{\varphi^3}(\hbar) = \sum_{n=0}^{\infty} \hbar^n (2n-1)!![x^{2n}]e^{\frac{x^3}{3!\hbar}} = \sum_{\text{cubic graphs } \Gamma} \frac{\hbar^{|E(\Gamma)|-|V(\Gamma)|}}{|\operatorname{Aut}\Gamma|}$$

- The parameter $\hbar$ counts the excess of the graph.



$$= 1 + \left(\frac{1}{8} + \frac{1}{12}\right)\hbar + \frac{385}{1152}\hbar^2 + \dots$$

where $\phi(\Gamma) = \hbar^{|E(\Gamma)|-|V(\Gamma)|}$ maps a graph with excess n to $\hbar^n$.

Arbitrary degree distribution

- More general with arbitrary degree distribution,

$$Z(\hbar) = \sum_{n=0}^{\infty} \hbar^n (2n-1)!![x^{2n}] e^{\frac{\sum_{k\geq 3} \frac{\lambda_k}{k!} x^k}{\hbar}}$$

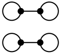
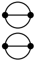
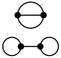
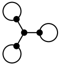
$$= \sum_{\text{graphs } \Gamma} \hbar^{|E(\Gamma)|-|V(\Gamma)|} \frac{\prod_{v\in V(\Gamma)} \lambda_{|v|}}{|\operatorname{Aut}\Gamma|}$$

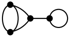
where the sum is over all multigraphs Γ and $|v|$ is the degree of vertex v.

- This corresponds to the pairing model of multigraphs [Bender and Canfield, 1978].

We obtain a series of polynomials

$$\sum_{n=0}^{\infty} \hbar^n (2n-1)!![x^{2n}] e^{\frac{\sum_{k\geq 3} \frac{\lambda_k}{k!} x^k}{\hbar}} = \phi\Big(1 + \frac{1}{8}$$  $$+ \frac{1}{12}$$  $$+ \frac{1}{8}$$ 

$$+ \frac{1}{128}$$  $$+ \frac{1}{288}$$  $$+ \frac{1}{96}$$  $$+ \frac{1}{48}$$ 

$$+ \frac{1}{16}$$  $$+ \frac{1}{16}$$  $$+ \frac{1}{8}$$  $$+ \frac{1}{24}$$  $$+ \cdots \Big)$$

where $\phi(\Gamma) = \hbar^{|E(\Gamma)|-|V(\Gamma)|} \prod_{v\in V(\Gamma)} \lambda_{|v|}$

$$= 1 + \left(\left(\frac{1}{8} + \frac{1}{12} \right) \lambda_3^2 + \frac{1}{8}\lambda_4 \right) \hbar$$

$$+ \left(\frac{385}{1152}\lambda_3^4 + \frac{35}{64}\lambda_3^2\lambda_4 + \frac{35}{384}\lambda_4^2 + \frac{7}{48}\lambda_3\lambda_5 + \frac{1}{48}\lambda_6 \right) \hbar^2 + \cdots$$

$$= \sum_{n=0}^{\infty} P_n(\lambda_3, \lambda_4, \ldots) \hbar^n$$

Example

Take the degree distribution $\lambda_k = -1$ for all $k \geq 3$.

$$Z^{\text{stir}}(\hbar) := \sum_{n=0}^{\infty} \hbar^n (2n-1)!! [x^{2n}] e^{\frac{\sum_{k\geq 3} \frac{-1}{k!} x^k}{\hbar}}$$

$= \phi\Big(1 + \frac{1}{8}$  $+ \frac{1}{12}$  $+ \frac{1}{8}$  $+ \dots\Big)$

where $\phi(\Gamma) = (-1)^{|V(\Gamma)|} \hbar^{|E(\Gamma)|-|V(\Gamma)|}$

$$= 1 + \left(\frac{1}{8}(-1)^2 + \frac{1}{12}(-1)^2 + \frac{1}{8}(-1)^1\right)\hbar + \dots$$

$$= 1 + \frac{1}{12}\hbar + \frac{1}{288}\hbar^2 - \frac{139}{51840}\hbar^3 + \dots = e^{\sum_{n=1}^{\infty} \frac{B_{n+1}}{n(n+1)} \hbar^n}$$

$\Rightarrow$ Stirling series as a signed sum over multigraphs.

- Define a functional $\mathcal{F} : \mathbb{R}[[x]] \to \mathbb{R}[[\hbar]]$

$$\mathcal{F} : \mathcal{S}(x) \mapsto \sum_{n=0}^{\infty} \hbar^n (2n-1)!![x^{2n}] e^{\frac{\frac{x^2}{2} + \mathcal{S}(x)}{\hbar}}$$

  where $\mathcal{S}(x) = -\frac{x^2}{2} + \sum_{k \geq 3} \frac{\lambda_k}{k!} x^k$.
- $\mathcal{F}$ maps a degree sequence to the corresponding generating function of multigraphs.

Expressions as

$$\mathcal{F}[\mathcal{S}(x)] = \sum_{n=0}^{\infty} \hbar^n (2n-1)!![x^{2n}] e^{\frac{\frac{x^2}{2}+\mathcal{S}(x)}{\hbar}}$$

are inconvenient to expand, because of the $\hbar$ and the $\frac{1}{\hbar}$ in the exponent $\Rightarrow$ we have to sum over a diagonal.

Theorem

This can be expressed without a diagonal summation [MB, 2017]:

$$\mathcal{F}[\mathcal{S}(x)] = \sum_{n=0}^{\infty} \hbar^n (2n+1)!![y^{2n+1}] x(y),$$

where $x(y)$ is the (power series) solution of

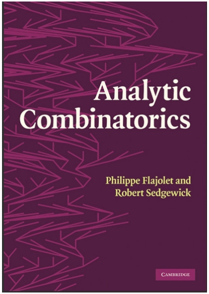
$$\frac{y^2}{2} = -\mathcal{S}(x(y)).$$

Asymptotics

Calculate asymptotics of $\mathcal{F}[\mathcal{S}(x)]$ by singularity analysis of $x(y)$:

$\Rightarrow$ Locate the dominant singularity of $x(y)$ by analysis of the (generalized) hyperelliptic curve,

$$\frac{y^2}{2} = -\mathcal{S}(x).$$



- The dominant singularity of $x(y)$ coincides with a branch-cut of the local parametrization of the curve.

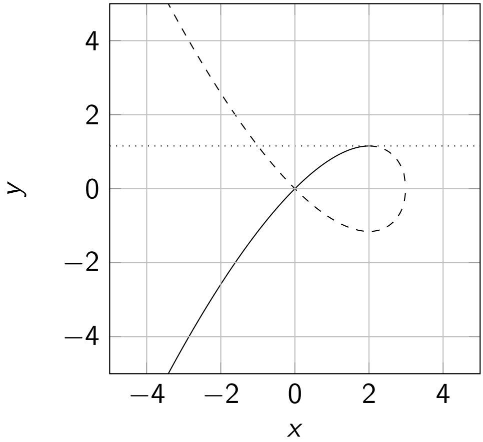


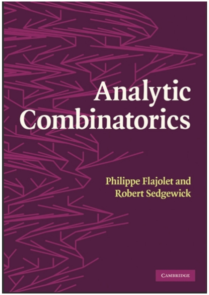
Figure: Example: The curve $\frac{y^2}{2} = \frac{x^2}{2} - \frac{x^3}{3!}$ associated to Z^{φ^3}.

$\Rightarrow x(y)$ has a (dominant) branch-cut singularity at $y = \rho = \frac{2}{\sqrt{3}}$, where $x(\rho) = \tau = 2$.

$\Rightarrow$ Near the dominant singularity:

$$x(y) - \tau = -d_1 \sqrt{1 - \frac{y}{\rho}} + \sum_{j=2}^{\infty} d_j \left(1 - \frac{y}{\rho}\right)^{\frac{j}{2}}$$

$$[y^n]x(y) \sim C \rho^{-n} n^{-\frac{3}{2}} \left(1 + \sum_{k=1}^{\infty} \frac{e_k}{n^k}\right)$$



- Therefore,

$$\begin{aligned}[\hbar^n]\mathcal{F}[\mathcal{S}(x)] &= (2n+1)!![y^{2n+1}]x(y) \\ &= \frac{2^n \Gamma(n + \frac{3}{2})}{\sqrt{\pi}} [y^{2n+1}]x(y) \\ &\sim C' 2^n \rho^{-2n} \Gamma(n) \left(1 + \sum_{k=1}^{\infty} \frac{e'_k}{n^k}\right)\end{aligned}$$

- What is the value of C' and the e'_k?

Theorem ([MB, 2017])

Let $(x,y) = (\tau,\rho)$ *be the location of the dominant branch-cut singularity of* $\frac{y^2}{2} = -\mathcal{S}(x)$*. Then*

$$[\hbar^n]\mathcal{F}[\mathcal{S}(x)](\hbar) = \sum_{k=0}^{R-1} c_k A^{-(n-k)}\Gamma(n-k) + \mathcal{O}\left(A^{-n}\Gamma(n-R)\right),$$

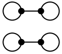
where $A = -\mathcal{S}(\tau)$ *and*

$$c_k = \frac{1}{2\pi}[\hbar^k]\mathcal{F}[\mathcal{S}(\tau) - \mathcal{S}(x+\tau)](-\hbar).$$

$\Rightarrow$ The asymptotic expansion can be expressed as a generating function of graphs.

Example

- For cubic graphs or equivalently φ^3 theory, we are interested in $\mathcal{S}(x) = -\frac{x^2}{2} + \frac{x^3}{3!}$,

$$\mathcal{F}\left[\mathcal{S}(x)\right](\hbar) = \phi\Big(1 + \frac{1}{8}$$  $$+ \frac{1}{12}$$  $$+ \frac{1}{128}$$  $$+ \ldots\Big)$$

$$1 + \frac{5}{24}\hbar + \frac{385}{1152}\hbar^2 + \frac{85085}{82944}\hbar^3 + \cdots$$

- We find $\tau = 2$, $A = \frac{2}{3}$ and the coefficients of the asymptotic expansion

$$\sum_{k=0}^{\infty} c_k \hbar^k = \frac{1}{2\pi}\mathcal{F}[\mathcal{S}(\tau) - \mathcal{S}(\tau + x)](-\hbar) = \frac{1}{2\pi}\mathcal{F}[-\frac{x^2}{2} + \frac{x^3}{3!}](-\hbar)$$

$$= \frac{1}{2\pi}\left(1 - \frac{5}{24}\hbar + \frac{385}{1152}\hbar^2 - \frac{85085}{82944}\hbar^3 + \ldots\right)$$

$\Rightarrow$ The asymptotic expansion is $[\hbar^n]\mathcal{F}\left[\mathcal{S}(x)\right](\hbar) = \sum_{k=0}^{R-1} c_k A^{-n+k}\Gamma(n-k) + \mathcal{O}(A^{-n+R}\Gamma(n-R))$.

Ring of factorially divergent power series

- Power series which have Poincaré asymptotic expansion of the form

$$f_n = \sum_{k=0}^{R-1} c_k A^{-n+k} \Gamma(n-k) + \mathcal{O}(A^{-n}\Gamma(n-R)) \quad \forall R \geq 0,$$

  form a subring of $\mathbb{R}[[x]]$ which is closed under composition and inversion of power series [MB, 2016].

- For instance, this allows us to calculate the complete asymptotic expansion of constructions such as

$$\log \mathcal{F}\left[\mathcal{S}(x)\right](\hbar)$$

  in closed form.

Renormalization

- Renormalization → Restriction on the allowed *bridgeless* subgraphs.
- P is a set of forbidden subgraphs.
- We wish to have a map

$$\phi_P(\Gamma) = \begin{cases} \hbar^{|E(\Gamma)|-|V(\Gamma)|} & \text{if } \Gamma \text{ has no subgraph in } P \\ 0 & \text{else} \end{cases}$$

- which is compatible with our generating function and asymptotic techniques.

- $\mathcal{G}$ is the $\mathbb{Q}$-algebra generated by multigraphs.
- Split $\mathcal{G} = \mathcal{G}^- \oplus \mathcal{G}^+$ such that
- $\mathcal{G}^+$ is the set of graphs **without** subgraphs in P.
- $\mathcal{G}^-$ is the set of graphs **with** subgraphs in P.
- We know a map $\phi : \mathcal{G} \to \mathbb{R}[[\hbar]]$.
- Construct ϕ_P such that $\phi_P|_{\mathcal{G}^+} = \phi$ and $\phi_P|_{\mathcal{G}^-} = 0$.
- This is a Riemann-Hilbert problem.

Hopf algebra of graphs

- Pick a set of bridgeless graphs P, such that

$$\text{if } \gamma_1 \subset \gamma_2 \text{ then } \gamma_1, \gamma_2 \in P \text{ iff } \gamma_1, \gamma_2/\gamma_1 \in P \tag{1}$$

$$\text{if } \gamma_1, \gamma_2 \in P \text{ then } \gamma_1 \cup \gamma_2 \in P \tag{2}$$

$$\emptyset \in P \tag{3}$$

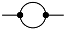
- $\mathcal{H}$ is the $\mathbb{Q}$-algebra of graphs in P.
- Define a coaction on $\mathcal{G}$:

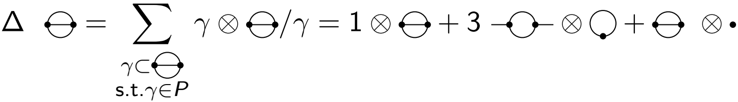
$$\begin{aligned} \Delta : \quad \mathcal{G} \quad &\rightarrow \quad \mathcal{H} \otimes \mathcal{G} \\ \Gamma \quad &\mapsto \sum_{\substack{\gamma \subset \Gamma \\ \text{s.t.} \gamma \in P}} \gamma \otimes \Gamma/\gamma \end{aligned}$$

- $\mathcal{H}$ is a left-comodule over $\mathcal{G}$.
- Δ can be set up on $\mathcal{H}$. $\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$.
- $\mathcal{H}$ is a Hopf algebra Connes and Kreimer [2001].
- (1) implies Δ is coassociative $(\text{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}) \circ \Delta$.

- The coaction shall keep information about the number of edges it was connected in the original graph.

$\Rightarrow$ The graphs in P have ‘legs’ or ‘hairs’.

- Example: Suppose $\emptyset$, ,  $\in P$, then



where we had to consider the subgraphs

the complete and the empty subgraph.

- Gives us an action on algebra morphisms $\mathcal{G} \to \mathbb{R}[[\hbar]]$. For $\psi : \mathcal{H} \to \mathbb{R}[[\hbar]]$ and $\phi : \mathcal{G} \to \mathbb{R}[[\hbar]]$,

$$\psi \star \phi : \mathcal{G} \to \mathbb{R}[[\hbar]] \qquad \psi \star \phi = m \circ (\psi \otimes \phi) \circ \Delta$$

- and a product of algebra morphisms $\mathcal{H} \to \mathbb{R}[[\hbar]]$. For $\xi : \mathcal{H} \to \mathbb{R}[[\hbar]]$ and $\psi : \mathcal{H} \to \mathbb{R}[[\hbar]]$,

$$\xi \star \psi : \mathcal{H} \to \mathbb{R}[[\hbar]] \qquad \xi \star \psi = m \circ (\xi \otimes \psi) \circ \Delta$$

- Coassociativity of Δ implies associativity of $\star$:

$$(\xi \star \psi) \star \phi = \xi \star (\psi \star \phi)$$

- The set of all algebra morphisms $\mathcal{H} \to \mathbb{R}[[\hbar]]$ with the $\star$-product forms a group.
- The identity $\epsilon : \mathcal{H} \to \mathbb{R}[[\hbar]]$ maps the empty graph to 1 and all other graphs to 0.
- The inverse of $\psi : \mathcal{H} \to \mathbb{R}[[\hbar]]$ maybe calculated recursively:

$$\psi^{\star -1}(\Gamma) = -\psi(\Gamma) - \sum_{\substack{\gamma \subsetneq \Gamma \\ \text{s.t.} \gamma \in P}} \psi^{\star -1}(\gamma)\psi(\Gamma/\gamma)$$

- Corresponds to Moebius-Inversion on the subgraph poset.
- Simplifies on many (physical) cases to a (functional) inversion problem on power series.
- Solves the Riemann-Hilbert problem:
  Invert $\phi : \Gamma \mapsto \hbar^{|E(\Gamma)|-|V(\Gamma)|}$ restricted to $\mathcal{H}$. Then

$$(\phi|_{\mathcal{H}}^{\star -1} \star \phi)(\Gamma) = \begin{cases} \hbar^{|E(\Gamma)|-|V(\Gamma)|} \text{ if } \Gamma \in \mathcal{G}^+ \\ 0 \text{ else} \end{cases}$$

The identity van Suijlekom [2007], Yeats [2008],

$$\Delta X = \sum_{\text{graphs } \Gamma} \left(\prod_{v \in V(\Gamma)} X_P^{(|v|)} \right) \otimes \frac{\Gamma}{|\operatorname{Aut} \Gamma|},$$

where $X = \sum_{\text{graphs } \Gamma} \frac{\Gamma}{|\operatorname{Aut} \Gamma|}$ and

$$X_P^{(k)} = \sum_{\substack{\Gamma \\ \text{s.t.} \Gamma \in P \text{ and} \\ \Gamma \text{ has } k \text{ legs}}} \frac{\Gamma}{|\operatorname{Aut} \Gamma|}.$$

can be used to make this accessible for asymptotic analysis:

$$\phi|_{\mathcal{H}}^{\star -1} \star \phi (X) = m \circ \left(\phi|_{\mathcal{H}}^{\star -1} \otimes \phi \right) \circ \Delta X$$

$$= \sum_{\text{graphs } \Gamma} \left(\prod_{v \in V(\Gamma)} \phi|_{\mathcal{H}}^{\star -1} \left(X_P^{(|v|)} \right) \right) \frac{\phi(\Gamma)}{|\operatorname{Aut} \Gamma|}$$

$$\phi|_{\mathcal{H}}^{\star-1} \star \phi(X) = \sum_{\text{graphs } \Gamma} \left(\prod_{v \in V(\Gamma)} \phi|_{\mathcal{H}}^{\star-1} \left(X_P^{(|v|)} \right) \right) \frac{\phi(\Gamma)}{|\operatorname{Aut}\Gamma|}$$

- The generating function of all graphs with arbitrary weight for each vertex degree is

$$\sum_{n=0}^{\infty} \hbar^n (2n-1)!![x^{2n}] e^{\frac{\sum_{k \geq 0} \frac{\lambda_k}{k!} x^k}{\hbar}}$$

$$= \sum_{\text{graphs } \Gamma} \left(\prod_{v \in V(\Gamma)} \lambda_{|v|} \right) \frac{\hbar^{|E(\Gamma)|-|V(\Gamma)|}}{|\operatorname{Aut}\Gamma|}$$

- Therefore,

$$\phi|_{\mathcal{H}}^{\star-1} \star \phi(X) = \sum_{n=0}^{\infty} \hbar^n (2n-1)!![x^{2n}] e^{\frac{\sum_{k \geq 0} \frac{x^k}{k!} \phi|_{\mathcal{H}}^{\star-1} \left(X_P^{(k)} \right)}{\hbar}}$$

$$\phi|_{\mathcal{H}}^{\star -1} \star \phi(X) = \sum_{n=0}^{\infty} \hbar^n (2n-1)!![x^{2n}] e^{\frac{\sum_{k\geq 0} \frac{x^k}{k!} \phi|_{\mathcal{H}}^{\star -1}\left(X_P^{(k)}\right)}{\hbar}}$$

- In quantum field theories we want the set P to be all graphs with a bounded number of legs.
- This corresponds to restrictions on the edge-connectivity of the graphs.
- In *renormalizable* quantum field theories the expressions $\phi|_{\mathcal{H}}^{\star -1}\left(X_P^{(k)}\right)$ are relatively easy to expand.
- The generating functions $\phi|_{\mathcal{H}}^{\star -1}\left(X_P^{(k)}\right)$ are called *counterterms* in this context.
- They are (almost) the generating functions of the number of *primitive* elements of $\mathcal{H}$:

$$\Gamma \text{ is primitive iff } \Delta\Gamma = \Gamma \otimes 1 + 1 \otimes \Gamma.$$

- We can obtain the full asymptotic expansions in these cases.

Examples

- Expressed as densities over all multigraphs:
- In φ^3-theory (i.e. three-valent multigraphs):
- $P(\Gamma \text{ is 2-edge-connected}) = e^{-1}\left(1 - \frac{23}{21}\frac{1}{n} + \ldots\right)$.
- $P(\Gamma \text{ is cyclically 4-edge-connected}) = e^{-\frac{10}{3}}\left(1 - \frac{133}{3}\frac{1}{n} + \cdots\right)$
- In φ^4-theory (i.e. four-valent multigraphs):
  $P(\Gamma \text{ is cyclically 6-edge-connected}) = e^{-\frac{15}{4}}\left(1 - 126\frac{1}{n} + \cdots\right)$.
- Arbitrary high order terms can be obtained by iteratively solving implicit equations.

Conclusions

- The 'zero-dimensional path integral' is a convenient tool to enumerate multigraphs by excess.
- Asymptotics are easily accessible: The asymptotic expansion also enumerates graphs.
- With Hopf algebra techniques restrictions on the set of enumerated graphs can be imposed.

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