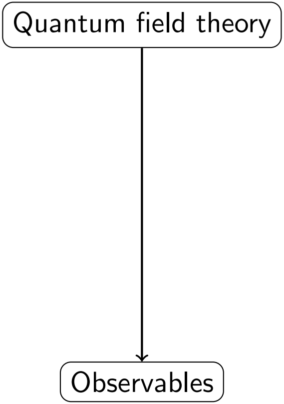
Graphs in perturbation theory: Algebraic structure and asymptotics

Michael Borinsky

Humboldt-University Berlin
Departments of Physics and Mathematics

Summer school on structures in local quantum field theory
Les Houches
6th June 2018

Motivation



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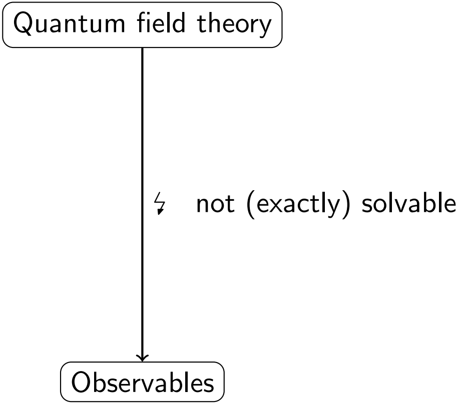

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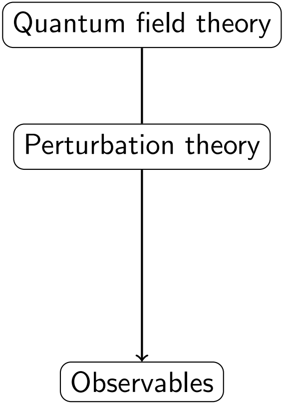

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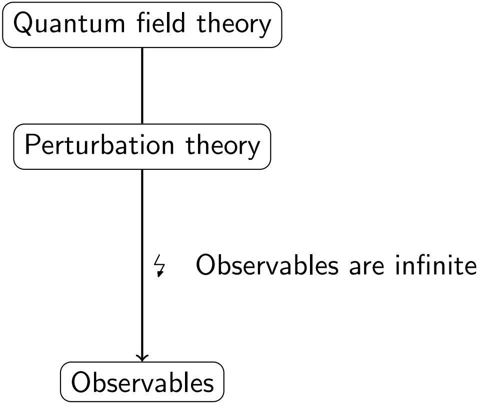
Quantum field theory
Perturbation theory
↯ Observables are infinite
Observables

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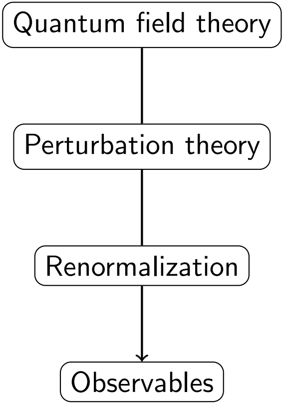
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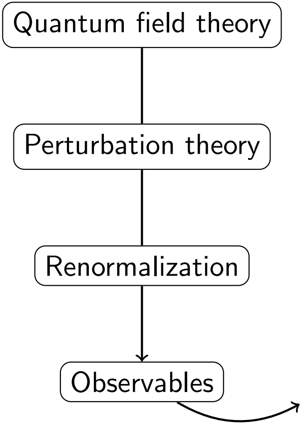

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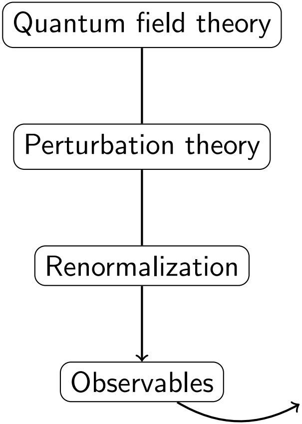

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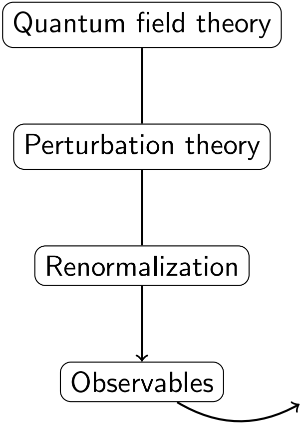

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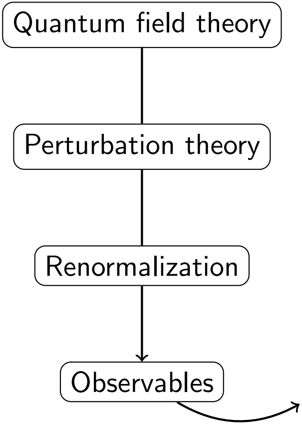
doable (pointing to αf_1)

Motivation



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hard

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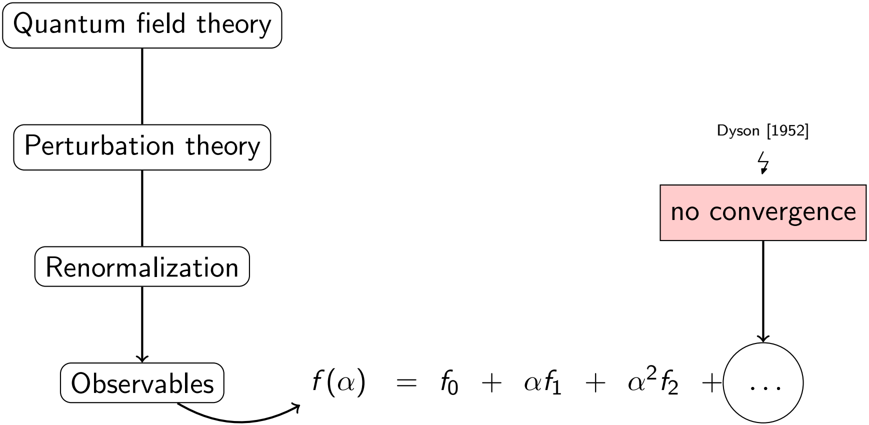
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Dyson [1952]
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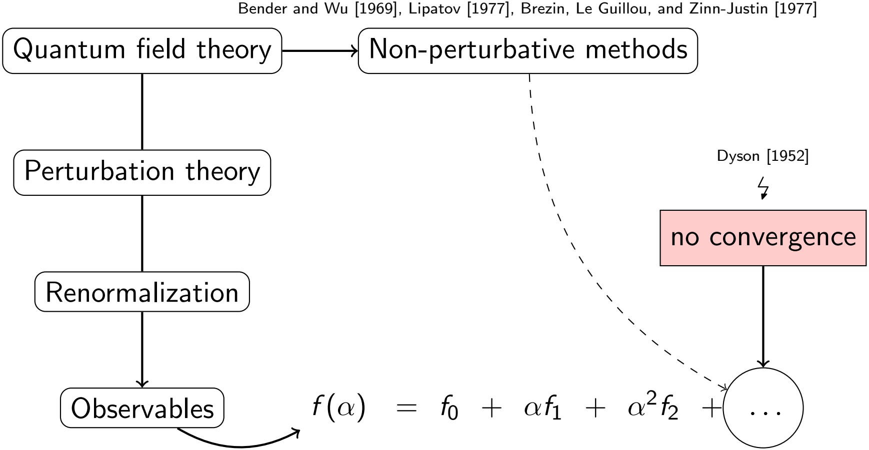

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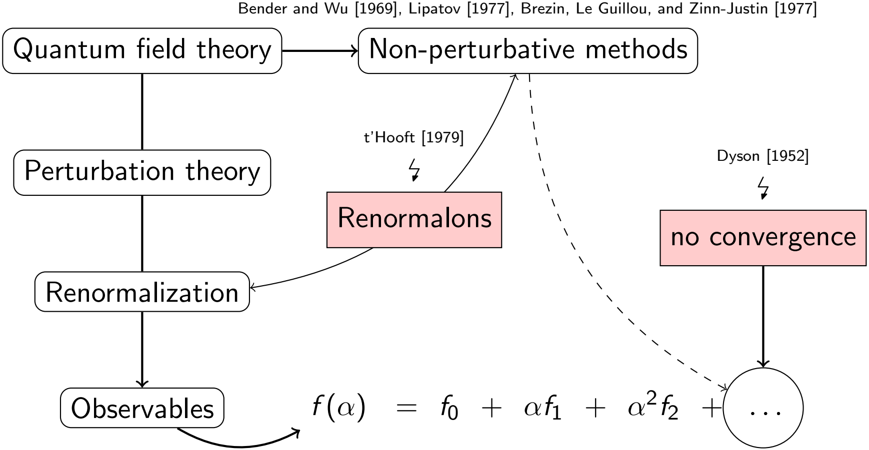
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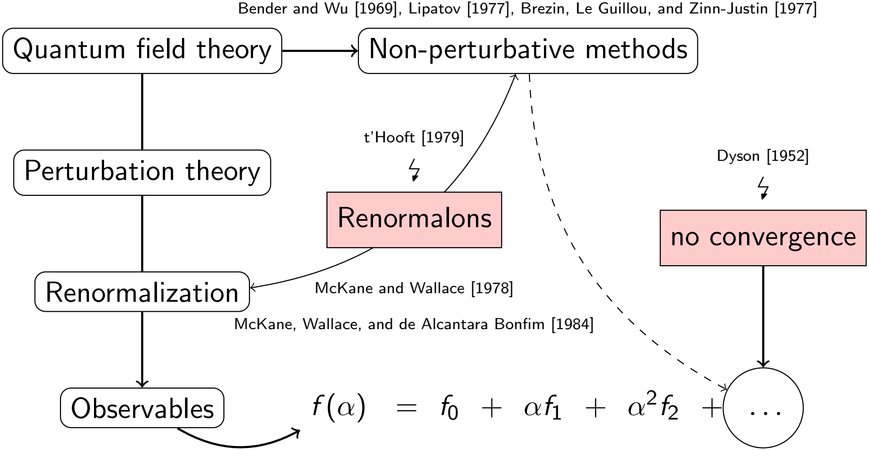
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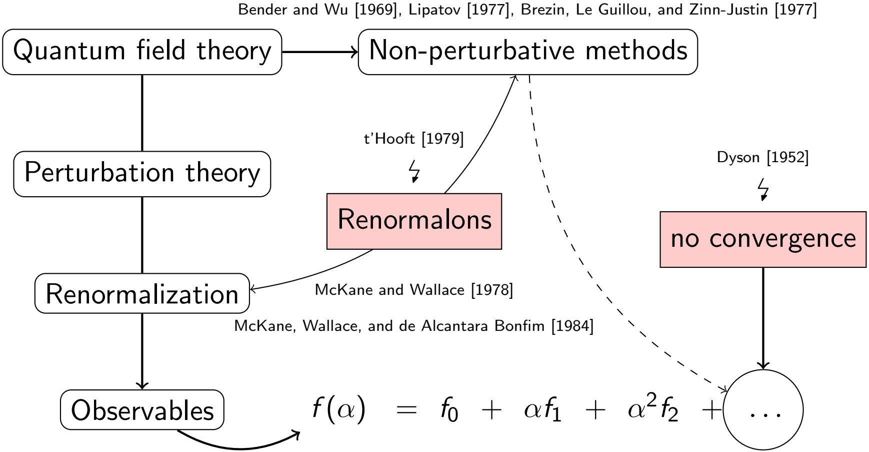

Large f_n inaccessible $\Leftrightarrow$ Large α inaccessible

An algebraic combinatorial study

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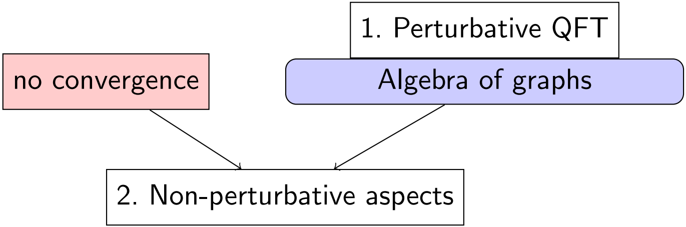
1. Perturbative QFT

Algebra of graphs

An algebraic combinatorial study



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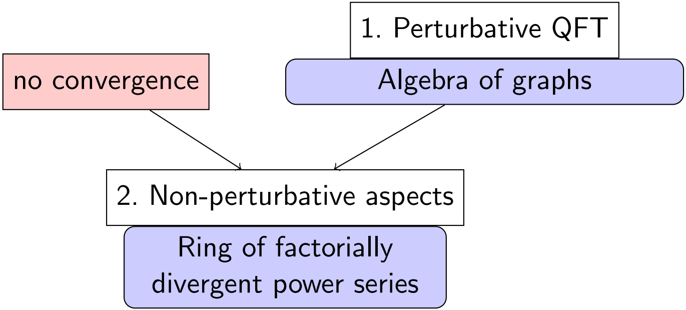

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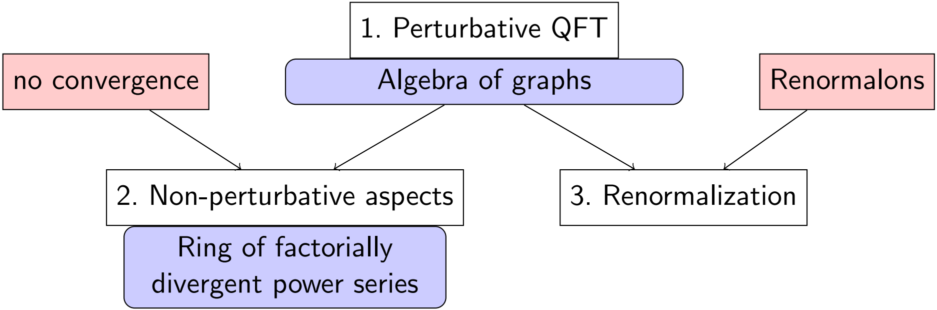

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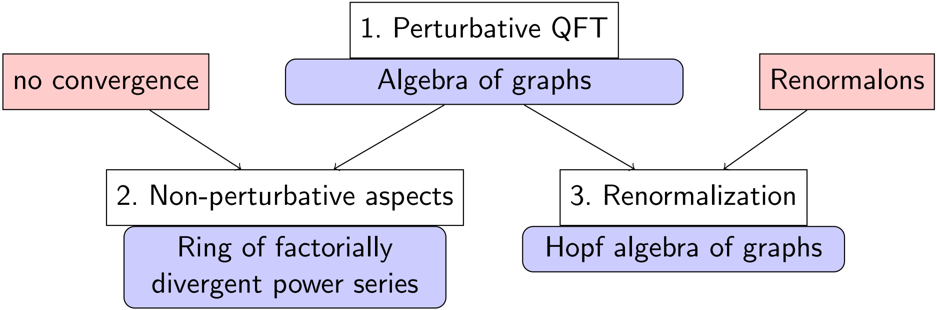

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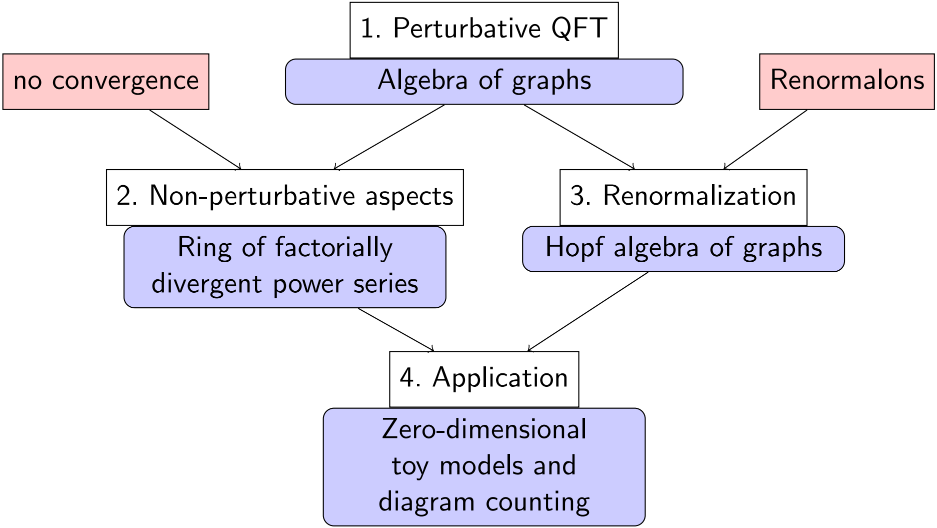

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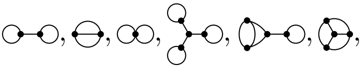
- Each graph represents an integral.

$\Rightarrow$ Use an algebra to represent graphs.

- Encode Feynman rules as **algebra homomorphisms**.

Algebra homomorphisms of graphs

The algebra of graphs:

$\mathcal{G} := \Big\langle \Big\{$  $, \ldots \Big\} \Big\rangle$

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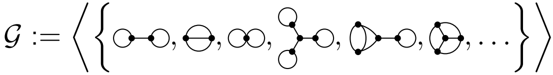

- **Feynman rules** are algebra homomorphisms $\phi : \mathcal{G} \to \mathbb{A}$.

- In zero-dimensional QFT:

$$\phi_{\boldsymbol{\lambda}} : \Gamma \mapsto \hbar^{\#\text{edges}-\#\text{vertices}} \prod_{v \in V_\Gamma} \lambda_{d_\Gamma^{(v)}},$$

where $d_\Gamma^{(v)}$ is the degree of the vertex v in Γ and the λ_k control the allowed degrees of the vertices.

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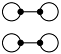
- Explicit access to unrenormalized quantities by path integral:

$$Z_{\boldsymbol{\lambda}}(\hbar) := \phi_{\boldsymbol{\lambda}} \left(\sum_{\text{graphs } \Gamma} \frac{\Gamma}{|\operatorname{Aut} \Gamma|} \right) = \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar}\left(-\frac{x^2}{2} + \sum_{k \geq 3} \lambda_k \frac{x^k}{k!}\right)}$$

$= \phi_{\boldsymbol{\lambda}} \Big(\mathbb{1} + \frac{1}{8}$  $+ \frac{1}{12}$  $+ \frac{1}{8}$  $+ \frac{1}{128}$  $+ \ldots \Big)$

$$= 1 + \left(\left(\frac{1}{8} + \frac{1}{12} \right) \lambda_3^2 + \frac{1}{8} \lambda_4 \right) \hbar + \ldots$$

Hurst [1952], Cvitanović, Lautrup, and Pearson [1978]

Argyres, van Hameren, Kleiss, and Papadopoulos [2001]

- Interpret observables as perturbation expansions

$$\int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar}\left(-\frac{x^2}{2}+\sum_{k\geq 3}\lambda_k \frac{x^k}{k!}\right)} = \sum_{n=0}^{\infty} z_n(\boldsymbol{\lambda})\hbar^n$$

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- The coefficients $z_n(\boldsymbol{\lambda})$ **count** graphs of excess n with degree distribution encoded in $\boldsymbol{\lambda}$.

- The large n asymptotics of $z_n(\boldsymbol{\lambda})$ are accessible

Theorem MB [2017]

$$z_n(\boldsymbol{\lambda}) \underset{n\to\infty}{=} A^{-n}\Gamma(n)\left(c_0(\boldsymbol{\lambda}) + c_1(\boldsymbol{\lambda})\frac{A}{n-1} + c_2(\boldsymbol{\lambda})\frac{A^2}{(n-1)(n-2)} + \ldots\right)$$

where with $\mathcal{S}(x) = -\frac{x^2}{2} + \sum_{k\geq 0} \lambda_k \frac{x^k}{k!}$

$$\int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{-\frac{1}{\hbar}(\mathcal{S}(x+\tau)-\mathcal{S}(\tau))} = \sum_{m=0}^{\infty} c_m(\boldsymbol{\lambda})(-\hbar)^m$$

and (τ, A) are the coordinates of the dominant saddle point of $\mathcal{S}(x)$, which can be obtained by analysis of the hyperelliptic curve $-\frac{y^2}{2} = \mathcal{S}(x)$.

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- This is a generalization of a result of Başar, Dunne, and Ünsal [2013] and a **resurgence** phenomenon.

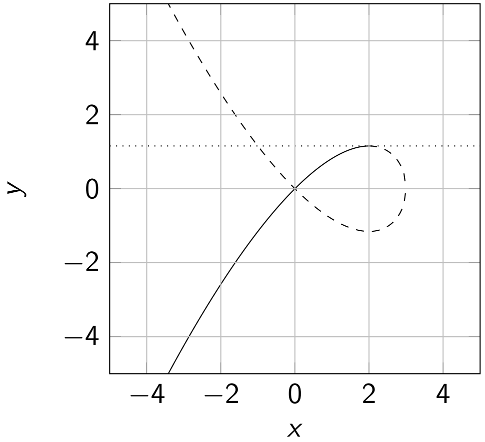
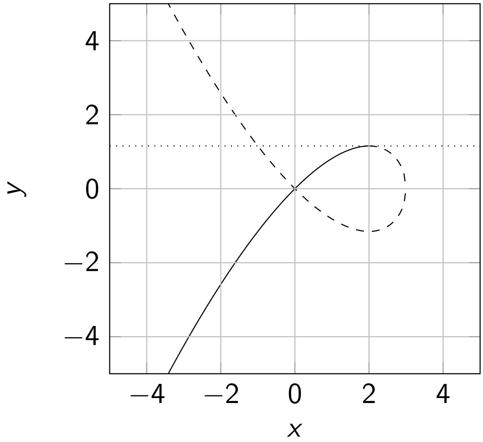


Figure: Example: The curve $\frac{y^2}{2} = \frac{x^2}{2} - \frac{x^3}{3!}$ associated to Z^{φ^3}.



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$\Rightarrow x(y)$ has a (dominant) branch-cut singularity at $y = \rho = \frac{2}{\sqrt{3}}$, where $x(\rho) = \tau = 2$.

Example

- For cubic graphs or equivalently φ^3 theory, we are interested in the action $-\frac{x^2}{2} + \frac{x^3}{3!}$, therefore $\lambda_3 = 1$ and $\lambda_k = 0$ for all $k \neq 3$,

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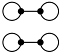
$\phi_{\boldsymbol{\lambda}_3}\Big(1 + \frac{1}{8}$  $+ \frac{1}{12}$  $+ \frac{1}{128}$  $+ \ldots\Big)$

$$1 + \frac{5}{24}\hbar + \frac{385}{1152}\hbar^2 + \frac{85085}{82944}\hbar^3 + \cdots$$

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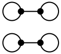
$\phi_{\boldsymbol{\lambda}_3}\big(1 + \frac{1}{8}$  $+ \frac{1}{12}$  $+ \frac{1}{128}$  $+ \ldots\big)$

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- We find $\tau = 2$, $A = \frac{2}{3}$ and the coefficients of the asymptotic expansion

$$\sum_{k=0}^{\infty} c_k(-\hbar)^k = \frac{1}{2\pi}\phi_{\boldsymbol{\lambda}_3'}\left(\sum_{\Gamma}\frac{\Gamma}{|\operatorname{Aut}\Gamma|}\right) = \frac{1}{2\pi}\phi_{\boldsymbol{\lambda}_3}\left(\sum_{\Gamma}\frac{\Gamma}{|\operatorname{Aut}\Gamma|}\right)$$

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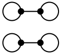
$\phi_{\boldsymbol{\lambda}_3}\Big(1 + \frac{1}{8}$  $+ \frac{1}{12}$  $+ \frac{1}{128}$  $+ \dots\Big)$

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$\Rightarrow$ The asymptotic expansion is $[\hbar^n]\mathcal{F}\left[\mathcal{S}(x)\right](\hbar) = \sum_{k=0}^{R-1} c_k A^{-n+k}\Gamma(n-k) + \mathcal{O}(A^{-n+R}\Gamma(n-R))$.

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- Asymptotic expansions can be extracted using the **ring of factorially divergent power series** MB [2016a].
- Powerseries version of **alien calculus** [Écalle, 1981].

Structure of factorially divergent power series

- Power series $\sum_{n\geq 0} f_n x^n$, which admit an asymptotic expansion

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- First step: Interpret the coefficients c_k as a new power series.
- Second step: Define an operator on $\mathbb{R}[[x]]^A$:

$$\begin{array}{ccccc} \mathcal{A} & : & \mathbb{R}[[x]]^A & \rightarrow & \mathbb{R}[[x]] \\ & & f(x) & \mapsto & \displaystyle\sum_{k\geq 0} c_k x^k \end{array}$$

- $\mathcal{A}$ is a derivation on $\mathbb{R}[[x]]^A$:

Theorem MB [2016a]

$$(\mathcal{A} f \cdot g)(x) = f(x)(\mathcal{A} g)(x) + (\mathcal{A} f)(x) g(x)$$

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Proof sketch

With $h(x) = f(x)g(x)$,

$$h_n = \underbrace{\sum_{k=0}^{R-1} f_{n-k} g_k + \sum_{k=0}^{R-1} f_k g_{n-k}}_{\text{High order times low order}} + \underbrace{\sum_{k=R}^{n-R} f_k g_{n-k}}_{\mathcal{O}(A^{-n}\Gamma(n-R))} .$$

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Theorem MB [2016a]

$$(\mathcal{A} f \cdot g)(x) = f(x)(\mathcal{A} g)(x) + (\mathcal{A} f)(x)g(x)$$

$\Rightarrow$ $\mathbb{R}[[x]]^A$ is a subring of $\mathbb{R}[[x]]$.

Proof sketch

With $h(x) = f(x)g(x)$,

$$h_n = \underbrace{\sum_{k=0}^{R-1} f_{n-k} g_k + \sum_{k=0}^{R-1} f_k g_{n-k}}_{\text{High order times low order}} + \underbrace{\sum_{k=R}^{n-R} f_k g_{n-k}}_{\mathcal{O}(A^{-n}\Gamma(n-R))} .$$

- $\sum_{k=R}^{n-R} f_k g_{n-k} \in \mathcal{O}(A^{-n}\Gamma(n-R))$ follows from the *log-convexity* of the Γ function.

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$$f \circ g \in \mathbb{R}[[x]]^A$$
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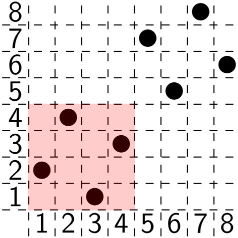
- Bender considered more general power series, but this is a direct corollary of his theorem in 1975.

Example

A reducible permutation:



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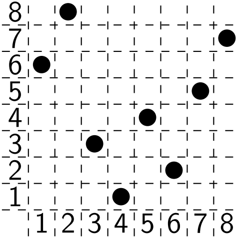

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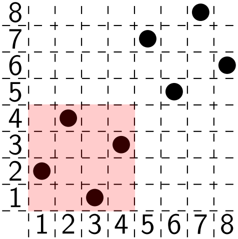

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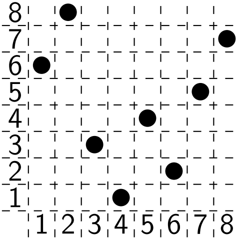

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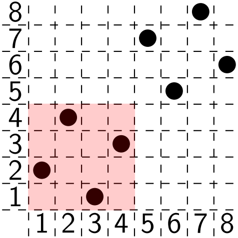

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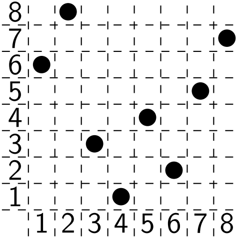

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Theorem Comtet [1972]

Therefore the asymptotic expansion of the coefficients of $I(x)$ is

$$[x^n]I(x) = \sum_{k=0}^{R-1} c_k (n-k)! + \mathcal{O}((n-R)!) \qquad \forall R \in \mathbb{N}_0,$$

where $c_k = [x^k]\frac{1}{(1+F(x))^2}$.

This chain rule can easily be generalized to multivalued analytic functions:

Theorem MB [2016a]

More general: For $f \in \mathbb{R}\{y_1, \ldots, y_L\}$ and $g^1, \ldots, g^L \in x\mathbb{R}[[x]]^A$:

$$(\mathcal{A}(f(g^1, \ldots, g^L))(x) = \sum_{l=1}^{L} \frac{\partial f}{\partial g^l}(g^1, \ldots, g^L)(\mathcal{A} g^l)(x).$$

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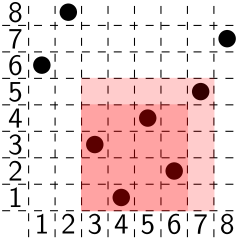
$\Rightarrow$ We can solve for asymptotics of implicitly defined power series.

Example: Simple permutations

A non-simple permutation:



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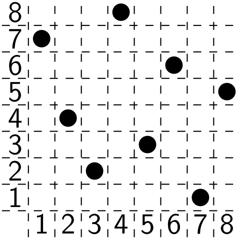

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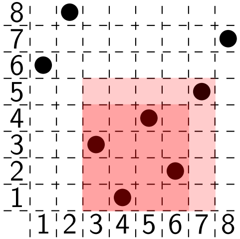

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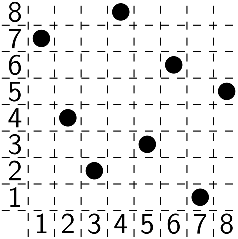

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- The OGF $S(x)$ of simple permutations fulfills

$$\frac{F(x) - F(x)^2}{1 + F(x)} = x + S(F(x)),$$

with $F(x) = \sum_{n=1}^{\infty} n! x^n$ [Albert, Klazar, and Atkinson, 2003].

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- Apply chain rule on both sides

$$\frac{1 - 2F(x) - F(x)^2}{(1 + F(x))^2}(\mathcal{A} F)(x) = S'(F(x))(\mathcal{A} F)(x) + \left(\frac{x}{F(x)}\right)^1 e^{\frac{F(x)-x}{xF(x)}}(\mathcal{A} S)(F(x)),$$

which can be solved for $(\mathcal{A} S)(x)$.

- After simplifications:

$$(\mathcal{A}\,S)(x) = \frac{1}{x}\frac{1}{1+x}\frac{1-x-(1+x)\frac{S(x)}{x}}{1+(1+x)\frac{S(x)}{x^2}}e^{-\frac{2+(1+x)\frac{S(x)}{x^2}}{1-x-(1+x)\frac{S(x)}{x}}}$$

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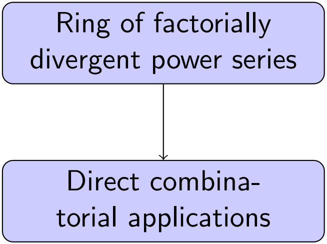
$$[x^n]S(x) = e^{-2} n! \left(1 - \frac{4}{n} + \frac{2}{n(n-1)} - \frac{40}{3n(n-1)(n-2)} + \dots\right),$$

the first three coefficients have been obtained by Albert, Klazar, and Atkinson [2003].

Ring of factorially divergent power series

- Allows to extract **explicit** asymptotic information from **implicitly** given power series.



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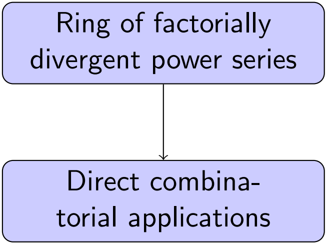

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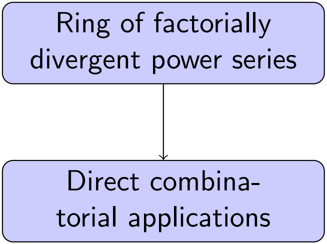

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- Generalized to allow arbitrary graphs.

- Starting point is to equip $\mathcal{G}$ with a **coproduct**:

$$\begin{array}{rccc} \Delta: & \mathcal{G} & \to & \mathcal{G} \otimes \mathcal{G} \\ & \Gamma & \mapsto & \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma \end{array}$$

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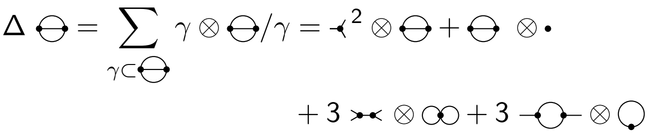
Example:



Hopf ideals in $\mathcal{G}$ MB [2018 PhD thesis]

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- The quotient of $\mathcal{G}/I_{\mathfrak{P}}$ with respect of one of these ideals is the Connes-Kreimer Hopf algebra.

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  is another algebra homomorphism.
- Every ideal $I_{\mathfrak{P}}$ gives rise to another group $\Phi_{\mathbb{A}}^{\mathcal{G}/I_{\mathfrak{P}}}$ which **acts** on $\Phi_{\mathbb{A}}^{\mathcal{G}}$.

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- Quotients $\mathcal{G}/I_{\mathfrak{P}}$ give rise to annihilation mappings,

$$\zeta^{\star -1}|_{\mathfrak{P}} \star \zeta(\Gamma) = \begin{cases} 1 & \text{if } \Gamma \text{ does not contain a subgraph from } \mathfrak{P}. \\ 0 & \text{else} \end{cases}$$

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- These maps allow us to obtain generating functions of graphs without subgraphs in $\mathfrak{P}$.

- We have an identity on $\mathcal{G}$ Kreimer [2006], van Suijlekom [2007], Yeats [2008]

$$\Delta \mathfrak{X} = \sum_{\Gamma} \prod_{v \in V_\Gamma} (d_\Gamma^{(v)}!) \mathfrak{X}^{(v)} \otimes \frac{\Gamma}{|\operatorname{Aut} \Gamma|},$$

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- The factors $\zeta^{\star -1}|_{\mathfrak{P}}\left(\mathfrak{X}^{(v)}\right)$ are the 'counterterms'.
- Explicit asymptotics can be obtained in the ring of factorially divergent power series.

Counting subgraph restricted graphs

- Let $f_{\mathfrak{M}}$ be the generating function of all graphs $\mathfrak{M}$ with marked degrees

$$f_{\mathfrak{M}}(\lambda_0, \lambda_1, \lambda_2, \ldots) = \sum_{\Gamma \in \mathfrak{M}} \frac{\prod_{v \in V_\Gamma} \lambda_{d_\Gamma^{(v)}}}{|\operatorname{Aut} \Gamma|} = \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2} + \sum_{k \geq 0} \lambda_k \frac{x^k}{k!}}$$

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$$sk_{\boldsymbol{\lambda}} : \Gamma \mapsto \begin{cases} \prod_{v \in V_\Gamma} \lambda_{d_\Gamma^{(v)}} & \text{if } \Gamma \text{ has no edges} \\ 0 & \text{else} \end{cases}$$

- Using the modified algebra homomorphism,

$$sk_{\boldsymbol{\lambda}} \star (\zeta^{\star -1}|_{\mathfrak{P}} \star \zeta)(\mathfrak{X}) = \sum_{\substack{\Gamma \in \mathfrak{M} \\ \text{s.t. } \Gamma \text{ has no subgraphs from } \mathfrak{P}}} \frac{\prod_{v \in V_\Gamma} \lambda_{d_\Gamma^{(v)}}}{|\operatorname{Aut}\Gamma|}$$

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gives the generating function

$$=: f_{\mathfrak{M}/\mathfrak{P}}(\lambda_0, \lambda_1, \dots)$$

of all graphs without subgraphs from $\mathfrak{P}$.

By using the factorization formula for the coproduct:

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where we expressed $f_{\mathfrak{M}/\mathfrak{P}}(\lambda_0, \lambda_1, \ldots)$ as a generalized composition of $f_{\mathfrak{M}}$ and $sk_{\boldsymbol{\lambda}} \star \zeta^{\star-1}|_{\mathfrak{P}}\left(\mathfrak{X}^{(k)}\right)$.

More explicitly

$$f_{\mathfrak{M}/\mathfrak{P}}(\lambda_0, \lambda_1, \ldots) = f_{\mathfrak{M}}\left((0!)g^0_{\mathfrak{P}}(\lambda_0, \ldots), (1!)g^1_{\mathfrak{P}}(\lambda_0, \ldots), \ldots\right)$$

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where

$$\begin{aligned} g^k_{\mathfrak{P}}(\lambda_0, \lambda_1, \ldots) &= sk_{\boldsymbol{\lambda}} \star \zeta^{\star -1}|_{\mathfrak{P}}\left(\mathfrak{X}^{(k)}\right) \\ &= \sum_{\substack{\Gamma \in \mathfrak{P} \\ \Gamma \text{ cntd. with } k \text{ legs}}} \zeta^{\star -1}|_{\mathfrak{P}}(\Gamma) \frac{\prod_{v \in V_\Gamma} \lambda_{d^{(v)}_\Gamma}}{|\operatorname{Aut} \Gamma|} \end{aligned}$$

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and $\zeta^{\star -1}|_{\mathfrak{P}}(\Gamma)$ can be expressed as a Moebius function,

$$\zeta^{\star -1}|_{\mathfrak{P}}(\Gamma) = -1 - \sum_{\substack{\gamma \subsetneq \Gamma \\ \gamma \in \mathfrak{P}}} \zeta^{\star -1}|_{\mathfrak{P}}(\gamma)$$

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- Using our results,

$$f_{\mathfrak{M}/\mathfrak{P}_{-\bullet}}(\lambda_0, \lambda_1, \ldots) = f_{\mathfrak{M}}\left((0!) g^{0}_{\mathfrak{P}_{-\bullet}}(\lambda_0, \ldots), (1!) g^{1}_{\mathfrak{P}_{-\bullet}}(\lambda_0, \ldots), \ldots \right)$$

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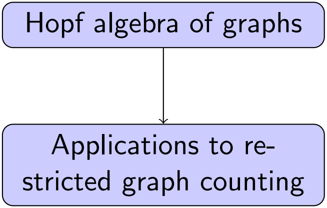
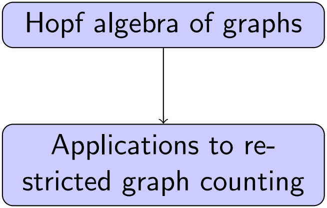
- Moreover, by analysing the Moebius function we find that

$$g^1_{\mathfrak{P}_{-\bullet}}(\lambda_0, \ldots) = - \sum_{\substack{\Gamma \in \mathfrak{P}_{-\bullet} \\ \text{s.t. } \Gamma \text{ is 1PI}}} \frac{\prod_{v \in V_\Gamma} \lambda_{d_\Gamma^{(v)}}}{|\operatorname{Aut} \Gamma|}$$

Hopf algebra of graphs



- Generating functions of subgraph restricted families of graphs can be obtained.



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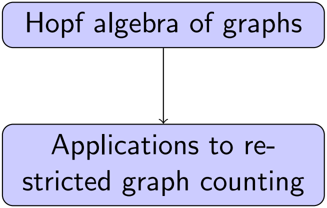

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- Feynman rules for physical theories carry additional structures MB [2016b].
- Hopf algebraic interpretation of the Legendre transformation in QFT MB [2018 PhD thesis].

Application

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Zero-dimensional toy models and diagram counting

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- Both Hopf algebra and factorially divergent power series may be used to study **zero-dimensional QFT** explicitly.
- All-order generating functions for asymptotics of renormalization quantities can be obtained. MB [2017]
- The densities of **primitive** diagrams can be computed.

Example

- The generating function of φ^4 primitives is

$$p(\hbar_{\text{ren}}) = 1 - z^{(\times)}(\hbar_{\text{ren}}) + 3 \sum_{n \geq 2} (-1)^n \left(\frac{\hbar_{\text{ren}}}{2} \right)^n$$

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- The asymptotics of this quantity can be obtained using the ring of factorially divergent power series MB [2017]:

$$[\hbar_{\text{ren}}^n] p(\hbar_{\text{ren}}) \underset{n \to \infty}{\sim} \frac{e^{-\frac{15}{4}}}{\sqrt{2}\pi} \left(\frac{2}{3} \right)^{n+3} \Gamma(n+3) \Big(36 + \\ - \frac{3}{2} \frac{243}{2} \frac{1}{n+2} + \left(\frac{3}{2} \right)^2 \frac{729}{32} \frac{1}{(n+1)(n+2)} + \ldots \Big).$$

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- Which can be compared with the expansion of the φ^4 β-function Kompaniets and Panzer [2017], where asymptotically only primitives are expected to contribute.

- Similarly, the number of primitives in quenched QED:

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$$\left. - \frac{4}{(2n-1)(2n+1)} - \frac{218}{3}\frac{1}{(2n-3)(2n-1)(2n+1)} + \ldots \right).$$

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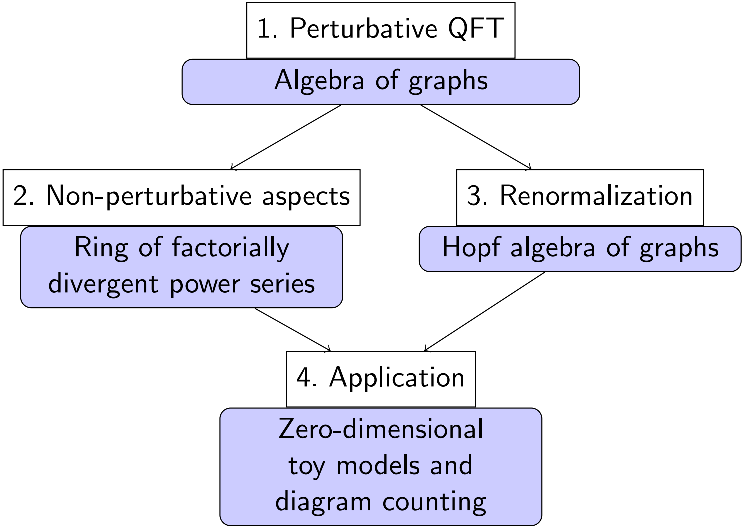
MB [2017] which resolves a question by David Broadhurst and Freeman Dyson.

Summary



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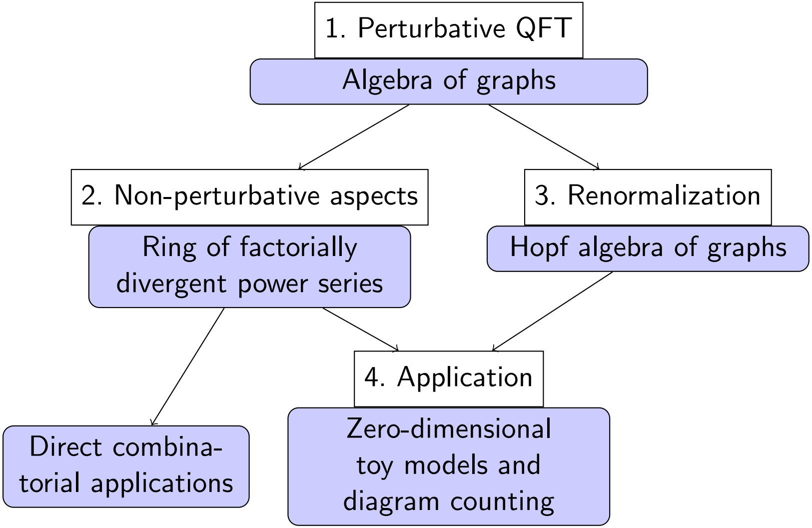
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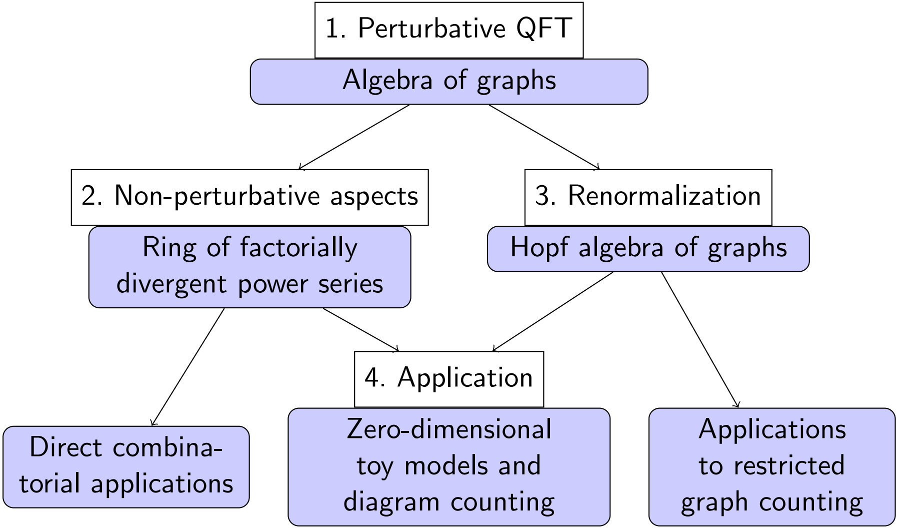
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Applications to restricted graph counting

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