Asymptotic Calculus for Combinatorial Dyson-Schwinger Equations

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For most systems, perturbation theory is necessary to compute physical quantities.

Often the perturbation expansions turn out to have vanishing radius of convergence!

Many of the expansions divergefactorially, i.e. 
\[ a_n \approx CA^n \Gamma(n + \beta) \] for large \( n \).

These expansions often have a combinatorial interpretation.

\[ \Rightarrow \] Analyse factorially divergent power series from a combinatorial perspective.

Treat factorially divergent power series analogous to the powerful framework of analytic combinatorics. Flajolet and Sedgewick [2009]
Suppose a power series behaves asymptotically as $A^n \Gamma(n + \beta)$ in contrast to, e.g. $e^{n^2}$, $\Gamma(\sqrt{n} + \beta)$, $\Gamma(n + \beta)^2$, etc.

In the $A^n \Gamma(n + \beta)$ case, knowledge of the asymptotic behaviour of one observable is enough to obtain knowledge of the asymptotic behaviour of all derived quantities.

This can be made quantitative by studying the ring of factorially divergent power series.
Factorially divergent power series

Consider the class of formal power series \( \mathbb{R}[[x]]^\alpha_\beta \subset \mathbb{R}[[x]] \) which admit an asymptotic expansion for large \( n \) of the form,

\[
f_n = \alpha^{n+\beta} \Gamma(n + \beta) \left( c_0 + \frac{c_1}{n + \beta} + \frac{c_2}{(n + \beta)(n + \beta - 1)} + \ldots \right)
\]

including power series with

\[
\lim_{n \to \infty} \frac{f_n}{\alpha^n \Gamma(n + \beta)} = 0
\]

\( \Rightarrow c_k = 0 \) for all \( k \geq 0 \).

Note, that the type of the asymptotic expansion is heavily restricted!
Consider a power series $f(x) \in \mathbb{R}[[x]]^\alpha_{\beta}$ for large $n$:

$$f_n = \alpha^{n+\beta} \Gamma(n + \beta) \left( c_0 + \frac{c_1}{n + \beta} + \frac{c_2}{(n + \beta)(n + \beta - 1)} + \ldots \right)$$

Idea: Interpret the coefficients $c_k$ of the asymptotic expansion as a new power series.

**Definition**

$\mathcal{A}$ maps a power series to its asymptotic expansion:

$$\mathcal{A} : \mathbb{R}[[x]]_{\beta}^\alpha \to \mathbb{R}[[x]]$$

$$f(x) \mapsto \gamma(x) = \sum_{k=0}^{\infty} c_k x^k$$
Theorem 1

$\mathcal{A}$ is a derivation on $\mathbb{R}[[x]]^{\alpha}_\beta$:

$$(\mathcal{A}f \cdot g)(x) = f(x)(\mathcal{A}g)(x) + (\mathcal{A}f)(x)g(x)$$

Follows from the log-convexity of $\Gamma$.

$\Rightarrow \mathbb{R}[[x]]^{\alpha}_\beta$ is a subring of $\mathbb{R}[[x]]$.

Proof sketch

With $h(x) = f(x)g(x)$,

$$h_n = \sum_{k=0}^{R-1} f_{n-k} g_k + \sum_{k=0}^{R-1} f_k g_{n-k} + \sum_{k=R}^{n-R} f_k g_{n-k}$$

High order times low order $\mathcal{O}(\alpha^n \Gamma(n+\beta-R))$.
What happens for composition of power series $\in \mathbb{R}[[x]]^\alpha_\beta$?

**Theorem 2 Bender [1975]**

If $|f_n| \leq C^n$ then, for $g \in \mathbb{R}[[x]]^\alpha_\beta$ with $g_0 = 0$:

$$f \circ g \in \mathbb{R}[[x]]^\alpha_\beta$$

$$(Af \circ g)(x) = f'(g(x))(Ag)(x)$$

Bender considered much more general power series, but this is a direct corollary of his theorem in 1975.
Theorem 3 MB [2016a]

More general for $f \in \mathbb{R}\{y_1, \ldots, y_L\}$ and $g^1, \ldots, g^L \in \mathbb{R}[[x]]^\alpha \beta$:

$$(A(f(g^1(x), \ldots, g^L(x))))(x) =$$

$$\sum_{l=1}^{L} \frac{\partial f}{\partial y_l}(y_1, \ldots, y_L) \bigg|_{y_m = g^m(x)} (A g^l)(x).$$

∀ $m \in \{1, \ldots, L\}$
What happens if $f \not\in \ker A$, i.e. $f$ does not have a finite radius of convergence.

$A$ fulfills a general ‘chain rule’:

**Theorem 4 MB [2016a]**

If $f, g \in \mathbb{R}[[x]]_{\alpha}^\beta$ with $g_0 = 0$ and $g_1 = 1$:

$$f \circ g \in \mathbb{R}[[x]]_{\alpha}^\beta$$

$$(Af \circ g)(x) = f'(g(x))(Ag)(x) + \left( \frac{x}{g(x)} \right)^\beta e^{\frac{g(x)-x}{\alpha xg(x)}} (Af)(g(x))$$

$\Rightarrow$ We can solve for asymptotics of implicitly defined power series.

- The factor $e^{\frac{g(x)-x}{\alpha xg(x)}}$ generates typical prefactors of the form

$$e^{\frac{g_2}{\alpha}}$$

in asymptotic expansions.
A chord diagram is the same as a single closed fermion loop with arbitrary photon interactions.

A connected diagram is the same as such a diagram without fermion self energy insertions.

Let \( I(x) = \sum_{n=0}^{\infty} (2n - 1)!! x^n \) be the ordinary generating function of all chord diagrams and \( C(x) \) the ordinary generating function of connected chord diagrams.

They are related by \( I(x) = 1 + C(xI(x)^2) \).
\[ I(x) = 1 + C(xI(x)^2) \]
\[ (AI)(x) = (AC(xI(x)^2))(x) \]
\[ (AI)(x) = 2xI(x)C'(xI(x)^2)(AI)(x) + \left( \frac{x}{xI(x)^2} \right)^{\frac{1}{2}} e^{\frac{xI(x)^2-x}{2x^2I(x)^2}} (AC)(xI(x)^2) \]

\[ I(x) \text{ is given by} \]
\[ I(x) = \sum_{n=0}^{\infty} (2n - 1)!!x^n \]
\[ = \sum_{n=0}^{\infty} \frac{2^n + \frac{1}{2}}{\sqrt{2\pi}} \Gamma\left(n + \frac{1}{2}\right)x^n \in \mathbb{R}[[x]]^{\frac{1}{2}} \]

- Using the chain rule for \( A \), we can solve for \( (AC)(x) \):
\[ (AC)(x) = \frac{1}{\sqrt{2\pi}} \frac{x}{C(x)} e^{-\frac{1}{2x}(2C(x)+C(x)^2)} \]
\[(\mathcal{A}\mathcal{C})(x) = \frac{1}{\sqrt{2\pi}} \frac{x}{C(x)} e^{-\frac{1}{2x}(2C(x)+C(x)^2)}\]

⇒ Generating function of the full asymptotic expansion of

\[C_n = (2n - 1)!! e^{-1} \left(1 - \frac{5}{2} \frac{1}{2n - 1} - \frac{43}{8} \frac{1}{(2n - 1)(2n - 3)} + \ldots\right) C_n = \]
Applications

Action on Dyson-Schwinger-Equations

Let $p, g, f \in \mathbb{R}[[x]]_{\alpha}^{\beta}$ and $p \in \ker \mathcal{A}$, then the functional equation,

$$p(g(x)) = x + f(g(x))$$

implies

$$(\mathcal{A}g)(x) = g'(x) \left( \frac{x}{g(x)} \right)^{\beta} e^{\frac{g(x)-x}{\alpha x g(x)}} (\mathcal{A}f)(g(x))$$

and

$$(\mathcal{A}f)(x) = g^{-1}'(x) \left( \frac{x}{g^{-1}(x)} \right)^{\beta} e^{\frac{g^{-1}(x)-x}{\alpha x g^{-1}(x)}} (\mathcal{A}g)(g^{-1}(x)).$$

where $g(g^{-1}(x)) = x$.

⇒ Solving the DSE ‘perturbativly’ to $n$ terms gives an asymptotic expansion up to order $n - 2$!

- $\mathcal{A}$ maps low order expansions to high order expansions.
- Asymptotic expansion independent of $p$. 
Example: Simple permutations

- Let $\pi \in S_n^{\text{simple}} \subset S_n$ such that $\pi([i,j]) \neq [k,l]$ for all $i,j,k,l \in [0, n]$ with $2 \leq |[i,j]| \leq n - 1$, then $\pi$ is a simple permutation, which does not map an interval to another interval.

- With $S(x) = \sum_{n=0}^{\infty} |S_n^{\text{simple}}| x^n$ and $F(x) = \sum_{n=1}^{\infty} n! x^n$:

  Albert et al. [2003]

  \[
  \frac{F(x) - F(x)^2}{1 + F(x)} = x + S(F(x))
  \]

- $F(x) \in \mathbb{R}[[x]]^1$ and $(\mathcal{A}F) = 1 \Rightarrow$ even though $S(x)$ is only given implicitly, we have an asymptotic expansion.
Generating function for asymptotic coefficients of $S(x)$:

$$(AS)(x) = \frac{1}{1 + x} \frac{1 - x - (1 + x) \frac{S(x)}{x}}{1 + (1 + x) \frac{S(x)}{x^2}} e^{-\frac{2+(1+x)\frac{S(x)}{x^2}}{1-x-(1+x)\frac{S(x)}{x}}}$$

$s_n = e^{-2} n! \left( 1 - 4 \frac{1}{n} + 2 \frac{1}{n(n-1)} - \frac{40}{3} \frac{1}{n(n-1)(n-2)} + \ldots \right)$

Generating function for asymptotic coefficients $\Rightarrow$ can analyze asymptotics of asymptotics.
The ring of factorially divergent power series

- \( \mathbb{R}[[x]]^\alpha_\beta \) forms a subring of \( \mathbb{R}[[x]] \) closed under multiplication, composition, differentiation and integration.

- \( \mathcal{A} \) is a derivation on \( \mathbb{R}[[x]]^\alpha_\beta \) which can be used to obtain asymptotic expansions of implicitly defined power series.

- Nice closure properties under asymptotic derivative \( \mathcal{A} \).

- Generalizations possible to multiple \( \alpha_1, \ldots, \alpha_l \in \mathbb{C} \) with \( |\alpha_i| = \alpha \).

- Question: Which classes of power series are closed under the operation of the asymptotic derivative?
Some power series closed under the ‘asymptotic’ derivative

A huge set of examples for factorially divergent power series is given by the following formal integral:

$$(1){\cal Z}(\hbar) = \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar}\left(-\frac{x^2}{2} + F(x)\right)}$$

This is to be interpreted as the power series given by,

$$ Z(\hbar) = \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{-\frac{x^2}{2\hbar}} x^n [y^n] e^{\frac{F(y)}{\hbar}} $$

$$ = \sum_{n=0}^{\infty} (2n-1)!! \hbar^n [y^{2n}] e^{\frac{F(y)}{\hbar}} $$

which gives a valid power series expansion in $\mathbb{R}[[\hbar]]$ for $F(x) \in x^3\mathbb{R}[[x]]$. 
Expansion of a zero-dimensional QFT. Cvitanović et al. [1978], Argyres et al. [2001], Hurst [1952], Molinari and Manini [2006]

Also the combinatorial generating function of the Feynman graphs contributing to the QFT with the interaction given by $F(x)$.

Maps from power series with non-vanishing radius of convergence to factorial growth power series.

Perfect ground to study the divergence of the perturbation expansion in general QFTs!
\[ Z(\hbar) = \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar} \left(-\frac{x^2}{2} + \sum_{k \geq 3} \lambda_k \frac{x^k}{k!}\right)} \]

- Combinatorial interpretation:

\[ Z(\hbar) = 1 + \frac{1}{8} \quad + \frac{1}{12} \quad + \frac{1}{8} \quad + \ldots \]

\[ = 1 + \hbar(\frac{1}{8} \lambda_3^2 \quad + \frac{1}{12} \lambda_3^2 \quad + \frac{1}{8} \lambda_4) \quad + \ldots \]

- \( Z \) counts graphs with weights \( \lambda \) assigned to each vertex. \( \hbar \) counts the **Euler characteristic** of the graph (i.e. \#loops − \#components)
Example

\[
Z^{\text{stir}}(\hbar) := \frac{\Gamma \left( \frac{1}{\hbar} \right)}{\sqrt{2\pi\hbar} \left( \frac{1}{\hbar} \right)^{\frac{1}{\hbar}}} \cdot e^{-\frac{1}{\hbar}} = \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar} \left( -\frac{x^2}{2} - (e^x - 1 - x - \frac{x^2}{2}) \right)}
\]

- **Combinatorial integral** representation of Stirling’s famous (asymptotic) expansion of the Gamma-function.

- Counts the (orbifold) Euler characteristic of the moduli space of (stable) open curves Kontsevich [1992],

\[
\log Z^{\text{stir}}(\hbar) = \sum_{g,n} \frac{\chi(\mathcal{M}_{g,n})}{n!} \hbar^{n+2g-2}
\]
Example

\[ Z^{\text{stir}}(\hbar) := \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar}\left(-\frac{x^2}{2} - (e^x - 1 - x - \frac{x^2}{2})\right)} \]

- Set \( F(x) = -(e^x - 1 - x - \frac{x^2}{2}) \). Combinatorial: All vertices are allowed and \( \lambda_k = -1 \).
- Diagrammatically:

\[ Z^{\text{stir}}(\hbar) = 1 + \frac{1}{8} + \frac{1}{12} + \frac{1}{8} + \cdots \]

\[ = 1 + \hbar \left( \frac{1}{8} (-1)^2 + \frac{1}{12} (-1)^2 + \frac{1}{8} (-1) \right) + \cdots \]

\[ = \frac{1}{12} \]

\[ = 1 + \hbar \frac{1}{12} + \hbar^2 \frac{1}{288} - \hbar^3 \frac{139}{51840} - \hbar^4 \frac{571}{2488320} + \cdots \]
\[ \mathcal{F}[F](\hbar) := \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar} \left(-\frac{x^2}{2} + F(x)\right)} = \sum_{n=0}^{\infty} (2n - 1)!! \hbar^n [y^{2n}] e^{\frac{F(y)}{\hbar}} \]

- Defines a map \( \mathcal{F} : x^3 \mathbb{R}[[x]] \to \mathbb{R}[[\hbar]]. \)
- Efficient calculation is possible using,

**Bivariate power series diagonalization**

\[
\int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar} \left(-\frac{x^2}{2} + F(x)\right)} = \\
\int_{\mathbb{R}} \frac{dy}{\sqrt{2\pi\hbar}} e^{-\frac{y^2}{2\hbar}} G'(y),
\]

where \( G(y) \) is the power series solution of \( \frac{y^2}{2} = \frac{G(y)^2}{2} - F(G(y)) \).
\[
\mathcal{F}[F](\hbar) = \sum_{n=0}^{\infty} (2n - 1)!![y^{2n}] G'(y)
\]

where \(G(y)\) is the (positive) solution of \(\frac{y^2}{2} = \frac{G(y)^2}{2} - F(G(y))\).

- The implicit equation \(\frac{y^2}{2} = \frac{G(y)^2}{2} - F(G(y))\) defines a complex curve in \(\mathbb{C}^2\).
- The asymptotics of \(\mathcal{F}[F](\hbar)\) is governed by the asymptotics of the convergent power series \(G(y)\).
- Asymptotics of \(G(y)\) can be calculated using methods of analytic combinatorics (Flajolet-Salvy algorithm). Banderier and Drmota [2015]
\[ \mathcal{F}[F](\hbar) := \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar} \left(-\frac{x^2}{2} + F(x)\right)} \]

- For minor restrictions on \( F \), the mapping
  \[ \mathcal{F} : x^3 \mathbb{R}[[x]] \rightarrow \mathbb{R}[[\hbar]] \]
  behaves nicely under the asymptotic derivative:

Asymptotics of combinatorial integrals MB [2016b]

\[ \mathcal{A}\mathcal{F}[F](\hbar) = \frac{1}{2\pi \sqrt{(F''(\tau) - 1)}} \mathcal{F}[\tilde{F}] \left(-\frac{\hbar}{F''(\tau) - 1}\right) \]

where \( \alpha = \frac{1}{\frac{\tau^2}{2} - F(\tau)} \), \( \beta = 0 \),

\[ \tilde{F}(x) = \frac{-F(x + \tau) - F(\tau) - xF'(\tau) - \frac{x^2}{2} F''(\tau)}{F''(\tau) - 1} \in x^3 \mathbb{R}[[x]] \]

and \( \tau \) is the dominant (branch cut) singularity associated to the curve

\[ \frac{y^2}{2} = \frac{x^2}{2} - F(x) . \]
$\mathcal{AF}[F](\hbar)$ is given by the expansion of the ‘combinatorial integral’ shifted to the ‘nearest’ saddle-point of the exponent,

$$Z(\hbar) = \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar}H(x)}$$

with $H(x) = -\frac{x^2}{2} + F(x)$.

As a mnemonic (not well-defined!)

$$AZ(\hbar) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar}(H(x+\tau)-H(\tau))}.$$

with $\tau$ the position of the ‘nearest’ saddle-point.

That means with $Z(\hbar) = \sum_{n=0} z_n \hbar^n$ and $\alpha = \frac{1}{H(\tau)}$

$$z_n =\sum_{k=0}^{R-1} \alpha^{n-k} \Gamma(n-k) [\hbar^k] AZ(\hbar) + O(\alpha^n \Gamma(n-R))$$
\[ \tilde{Z}^{\text{QED}}(\hbar) = \int_{\mathbb{R}} \frac{dA}{\sqrt{2\pi\hbar}} e^{-\frac{1}{\hbar} \sin^2(A)} \]

Using the formalism we see that \( \tilde{Z}^{\text{QED}} \in \mathbb{R}[[\hbar]]_{0}^{2} \) and
\[ A \tilde{Z}^{\text{QED}}(\hbar) = \frac{2}{2\pi} \tilde{Z}^{\text{QED}}(-\hbar) \]

Asymptotics of QED diagram counting MB [2016b]

Therefore with \( \tilde{Z}^{\text{QED}}(\hbar) = \sum_{n=0} z^{\text{QED}}_{n} \hbar^{n} \)
\[ z^{\text{QED}}_{n} = \frac{1}{\pi} \sum_{k=0}^{R-1} 2^{n-k} \Gamma(n-k)[\hbar^{k}] \tilde{Z}^{\text{QED}}(-\hbar) + O(2^{n} \Gamma(n-R)) \]

Full asymptotic expansion of \( \tilde{Z}^{\text{QED}}(\hbar) \).
The computation of the asymptotic expansion is as ‘easy’ as the computation of the original expansion.

From $\mathcal{A}\tilde{Z}^{\text{QED}}(\hbar)$ asymptotic expansions of all derived quantities can be obtained using the algebraic properties of the ring of factorially divergent power series.

Example:

$$\mathcal{A}\log \tilde{Z}^{\text{QED}}(\hbar) = \frac{\mathcal{A}\tilde{Z}^{\text{QED}}(\hbar)}{\tilde{Z}^{\text{QED}}(\hbar)} = \frac{1}{\pi} \frac{\tilde{Z}^{\text{QED}}(-\hbar)}{\tilde{Z}^{\text{QED}}(\hbar)}$$

This is the generating function of the asymptotic expansion of connected QED diagrams.

Implicit functional relations can be solved using the generalized chain rule.

This gives rise to the asymptotic expansions of ‘renormalized’ quantities. Combinatorially, these correspond to the number of skeleton or primitive diagrams.
Combinatorial integrals

\[ \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar} \left( -\frac{x^2}{2} + F(x) \right)} \]

with minor restrictions on \( F(x) \) provide a large set of generating functions which are algebraically closed under composition, inversion, and the asymptotic derivative.

Asymptotic expansions of arbitrary order can be obtained from a combinatorial integral as well as any implicitly given function of them.
Conclusions

- A direct application of the ring of factorially divergent power series is bringing the classical treatments of zero-dimensional QFT to the asymptotic level.
- Applications to graph-enumeration.
- ‘Canonical’ nature of combinatorial integrals, because of the resemblance to path integral formulations.
Further applications in QFT

- The divergence of the perturbation expansions in physical QFTs is believed to be governed by the growth of diagrams.
- In fact there are strong indications for this.
- Possible to give bounds on Feynman integrals at each loop order.

⇒ Formulate combinatorial models which encode these bounds in terms of combinatorial integrals and study their asymptotics.


CA Hurst. The enumeration of graphs in the feynman-dyson technique. In *Proceedings of the Royal Society of London A*:

