Tropicalized Quantum Field Theory

Michigan Interdisciplinary Meeting on Amplitudes

arXiv:2508.14263

Michael Borinsky — Perimeter Institute for Theoretical Physics

Slogan

Quantum Field theory is the best theory

One of physics most precise tools

PHYSICAL REVIEW LETTERS **130,** 071801 (2023)

Editors' Suggestion   Featured in Physics

Measurement of the Electron Magnetic Moment

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The electron magnetic moment, $-\mu/\mu_B = g/2 = 1.001\ 159\,652\ 180\,59\,(13)$ [0.13 ppt], is determined 2.2 times more accurately than the value that stood for fourteen years. The most precisely determined property of an elementary particle tests the most precise prediction of the standard model (SM) to 1 part in 10^{12}. The test would improve an order of magnitude if the uncertainty from discrepant measurements of the fine structure constant α is eliminated since the SM prediction is a function of α. The new measurement and SM theory together predict $\alpha^{-1} = 137.035\,999\,166\,(15)$ [0.11 ppb] with an uncertainty 10 times smaller than the current disagreement between measured α values.

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Amplitudes

$$\mathscr{A}_{L,n}(p_1,\ldots,p_n) = \sum_G \frac{1}{|\mathrm{Aut}(G)|} \int_{\mathbb{R}_{>0}^{E_G}} \frac{d\alpha_1 \ldots d\alpha_n}{\mathscr{P}_G(\boldsymbol{\alpha},\boldsymbol{p})^{D/2}}$$

1PI correlators / Amplitudes

Sum over graphs with L loops and n legs

Symmetry factor

Feynman integral: Rational integrand, over positive orthand.

Many shapes of Feynman integrals

$$\int_{\mathbb{R}_{>0}^{E_G}} \frac{d\alpha_1 \ldots d\alpha_n}{\mathscr{P}_G(\alpha, p)^{D/2}} = \star \int_{\mathbb{R}_{>0}^{E_G}} \exp(-F/U) \frac{d\alpha_1 \ldots d\alpha_n}{U^{D/2}} = \star \int_{\mathbb{M}^{DL}} \frac{d^D k_1 \cdots d^D k_L}{\prod_e (q_e^2 - m_e^2 + i\varepsilon)}$$

Here: Lee—Pomeransky polynomial/representation

$\mathscr{P}_G(\alpha, p)$ is the base generating polynomial of a matroid associated with G.

$$\mathscr{A}_{L,n}(p_1,\ldots,p_n) = \sum_G \frac{1}{|\mathrm{Aut}G|} \int_{\mathbb{R}^{E_G}_{>0}} \frac{d\alpha_1 \ldots d\alpha_n}{\mathscr{P}_G(\boldsymbol{\alpha},\boldsymbol{p})^{D/2}}$$

1PI correlators / Amplitudes

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Major problem: The number of graphs grows as $(L!)$

$$\mathscr{A}_{L,n}(p_1,\ldots,p_n) = \sum_G \frac{1}{|\mathrm{Aut}G|} \int_{\mathbb{R}_{>0}^{E_G}} \frac{d\alpha_1 \ldots d\alpha_n}{\mathscr{P}_G(\boldsymbol{\alpha},\boldsymbol{p})^{D/2}}$$

Major problem: The number of graphs grows as $(L!)$

Predictions take factorial computer-time
(and become more expensive than measurement)

Quantum field theory is also the worst theory.

It is very hard to actually make predictions.

$$\mathscr{A}_{L,n}(p_1, \ldots, p_n) = \sum_G \frac{1}{|\mathrm{Aut}G|} \int_{\mathbb{R}_{>0}^{E_G}} \frac{d\alpha_1 \ldots d\alpha_n}{\mathscr{P}_G(\boldsymbol{\alpha}, \boldsymbol{p})^{D/2}}$$

Key idea to overcome this: **Avoid individual Feynman graphs (somehow)**

Britto—Cachazo—Feng—Witten 2005

Arkani-Hamed—Trnka 2013

Arkani-Hamed—Bai—Lam 2017

New way of avoiding individual Feynman integrals:

Tropicalized QFT

arXiv:2508.14263

Relation to previous work on **tropicalized Feynman integrals**

Panzer 2019, MB 2020, MB—Munch—Tellander 2023

Inspiration from and relations to **Surfaceology**

Arkani-Hamed, Figueiredo, Frost, Salvatori, Plamondon, Thomas, … 2023—

Particularly

Salvatori 2025

Important empirical data:

Balduf 2023, Balduf—Shaban 2024, MB-Favorito 2025, Balduf—Thürigen 2025

$$\mathscr{A}_{L,n}(p_1, \ldots, p_n) = \sum_G \frac{1}{|\,\mathrm{Aut}G\,|} \int_{\mathbb{R}^{E_G}_{>0}} \frac{d\alpha_1 \ldots d\alpha_n}{\mathscr{P}_G(\boldsymbol{\alpha}, \boldsymbol{p})^{D/2}}$$

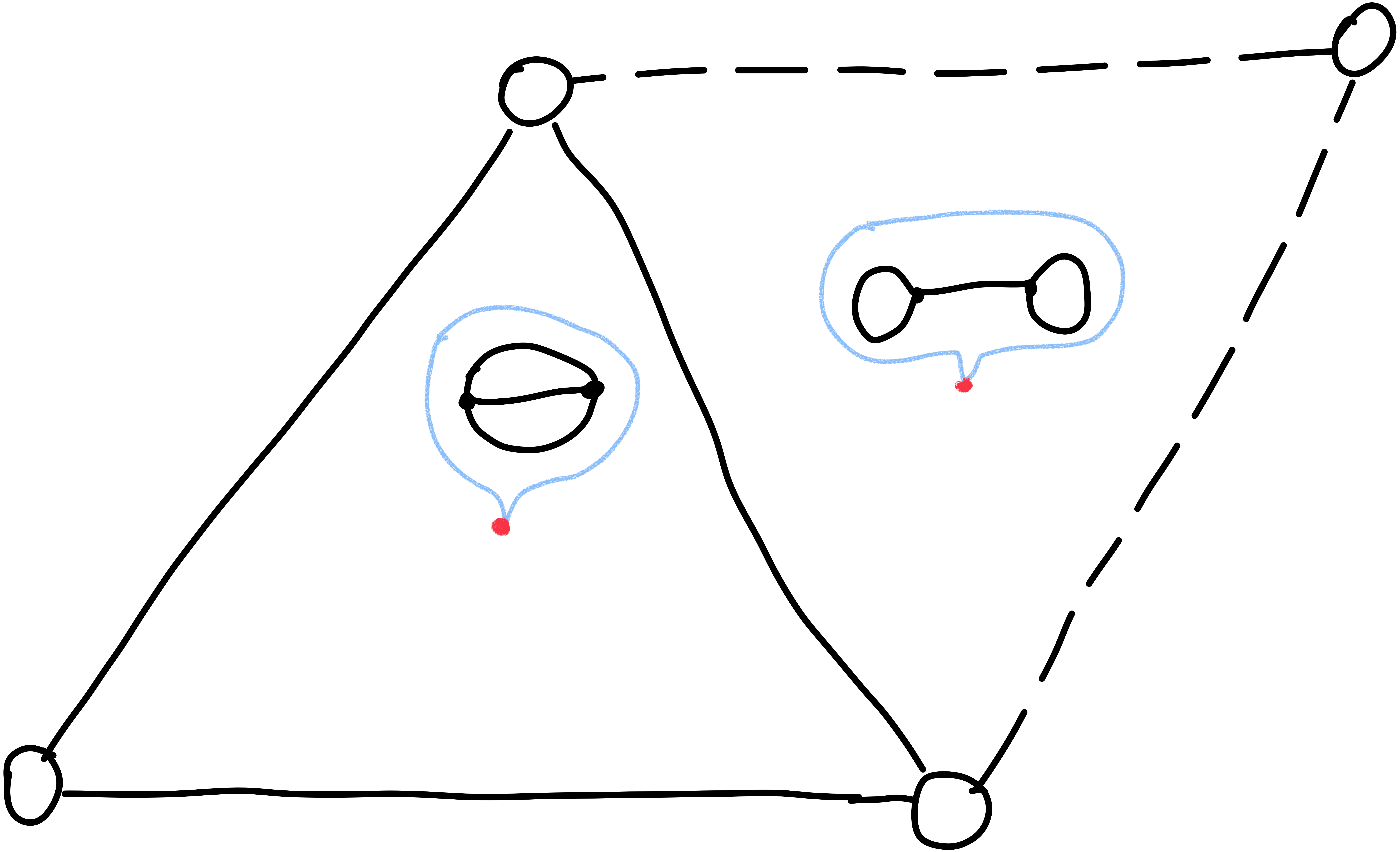
$$\mathscr{A}_{L,n}(p_1, \ldots, p_n) = \underbrace{\sum_G \frac{1}{|\mathrm{Aut}G|} \int_{\mathbb{R}^{E_G}_{>0}}}_{\text{Integral over a moduli space of graphs.}} \frac{d\alpha_1 \ldots d\alpha_n}{\mathscr{P}_G(\boldsymbol{\alpha}, \boldsymbol{p})^{D/2}}$$

$$
\begin{aligned}
\mathscr{A}_{L,n}(p_1,\ldots,p_n) &= \sum_G \frac{1}{|\mathrm{Aut}G|} \int_{\mathbb{R}_{>0}^{E_G}} \frac{d\alpha_1\ldots d\alpha_n}{\mathscr{P}_G(\boldsymbol{\alpha},\boldsymbol{p})^{D/2}} \\
&= \int_{\mathscr{M}_{L,n}^{\mathrm{tr}}} \mu_D(\boldsymbol{p})
\end{aligned}
$$

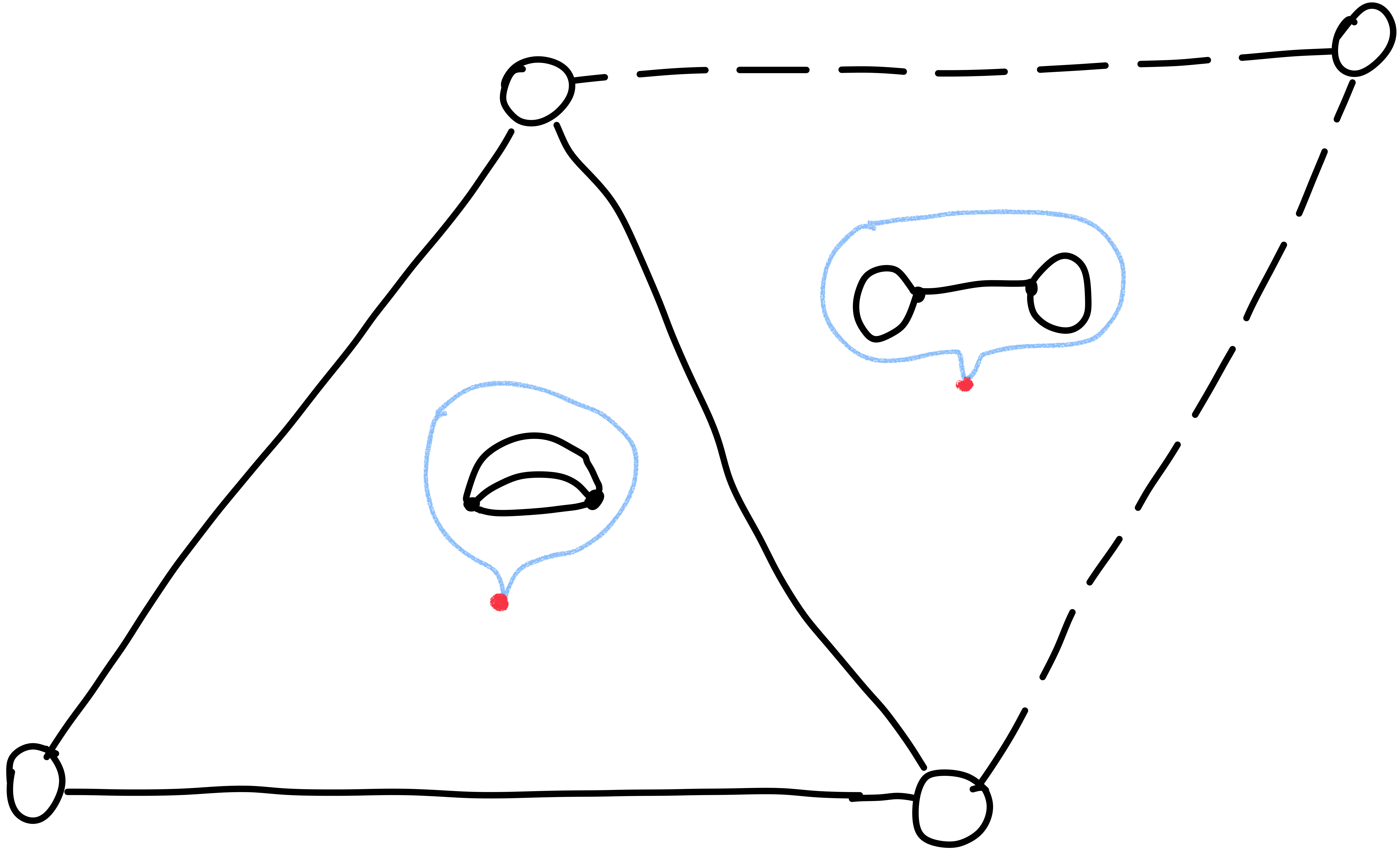
where $\quad \mathscr{M}_{L,n}^{\mathrm{tr}} = \bigsqcup_G \left(\mathbb{R}_{>0}^{E_G} \,/\, \mathrm{Aut}(G)\right)$

is (almost) the moduli space of tropical curves and the moduli space of graphs.

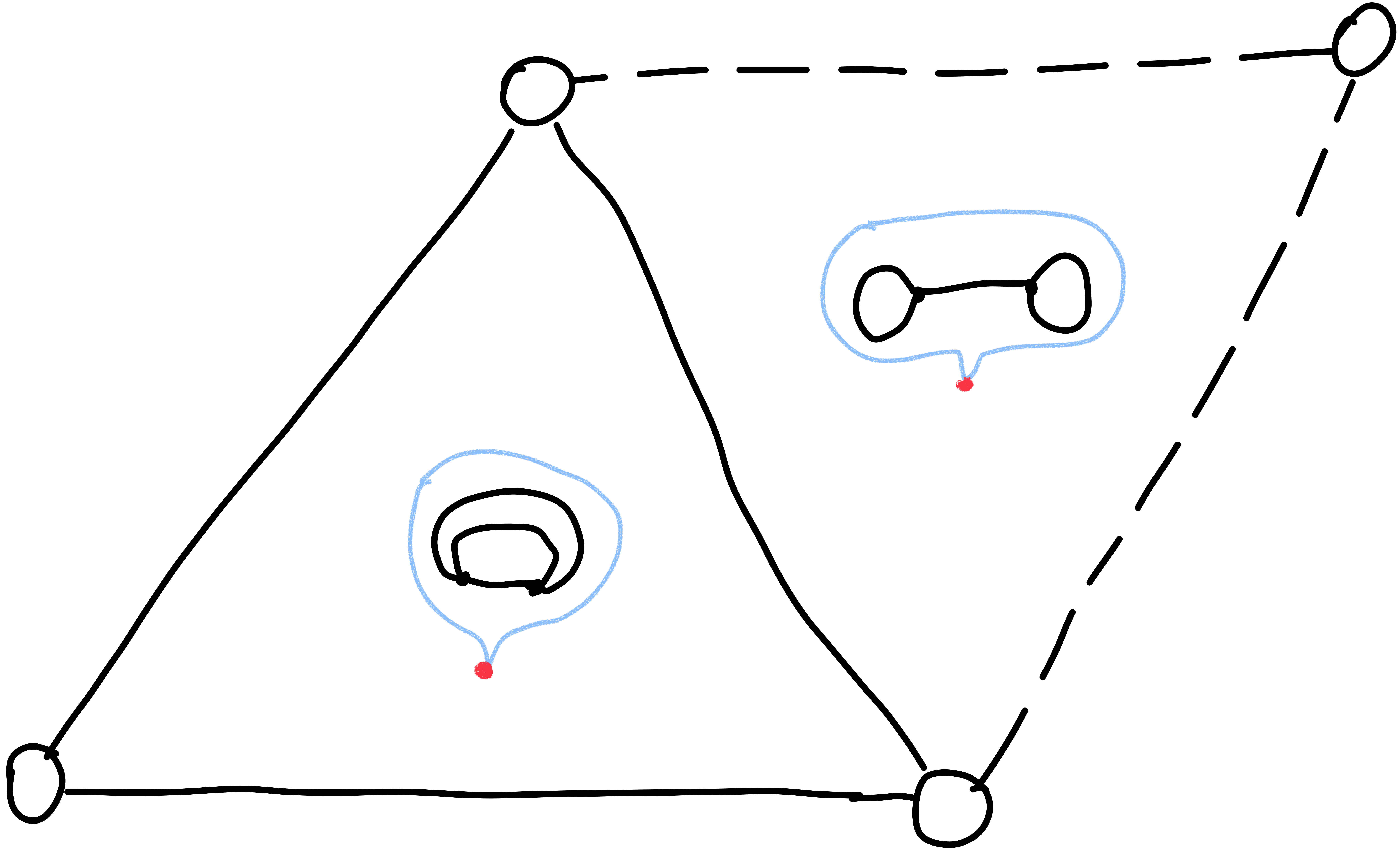
$\mathscr{M}^{\mathrm{tr}}_{2,0}$



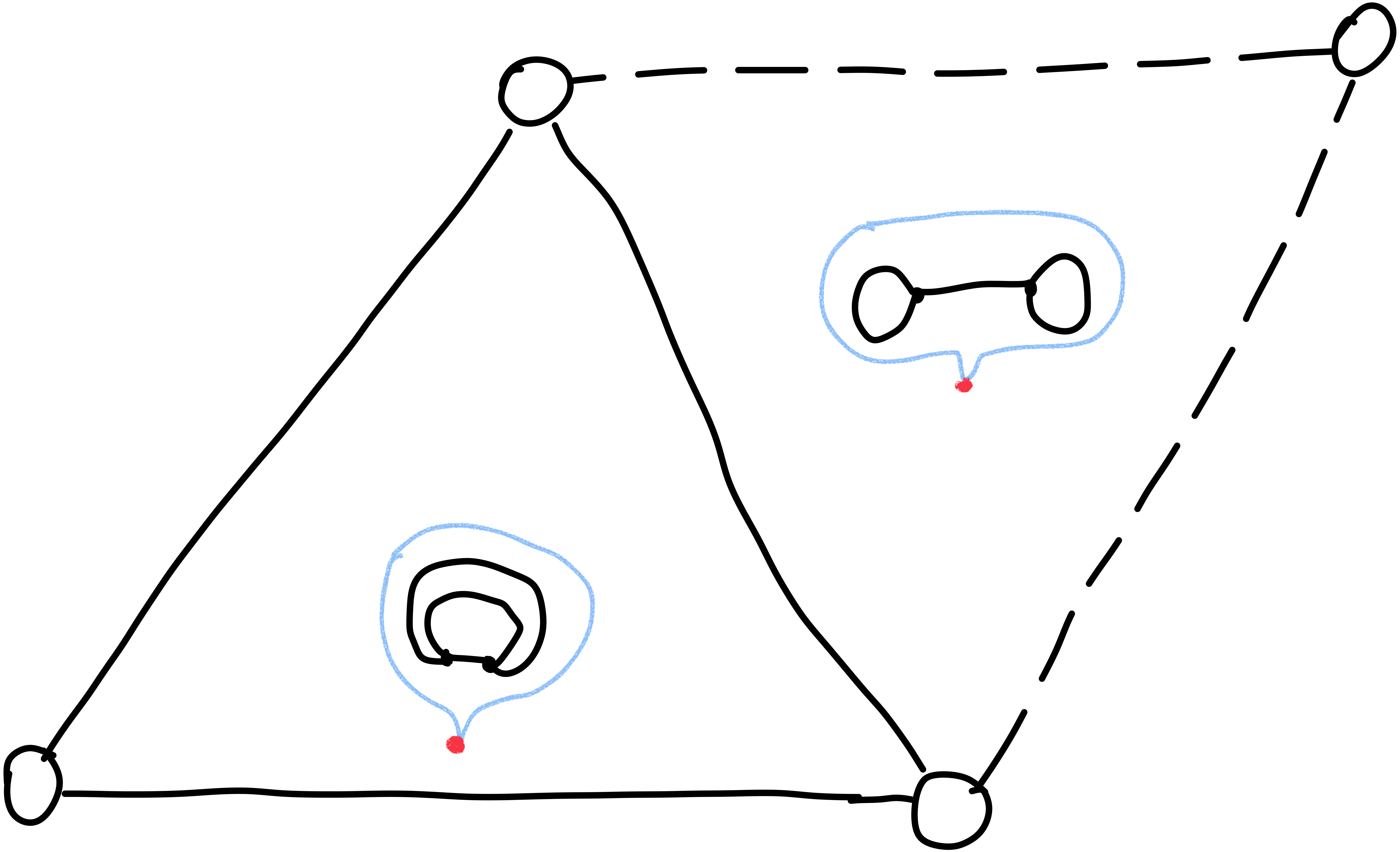
$\mathscr{M}^{\mathrm{tr}}_{2,0}$



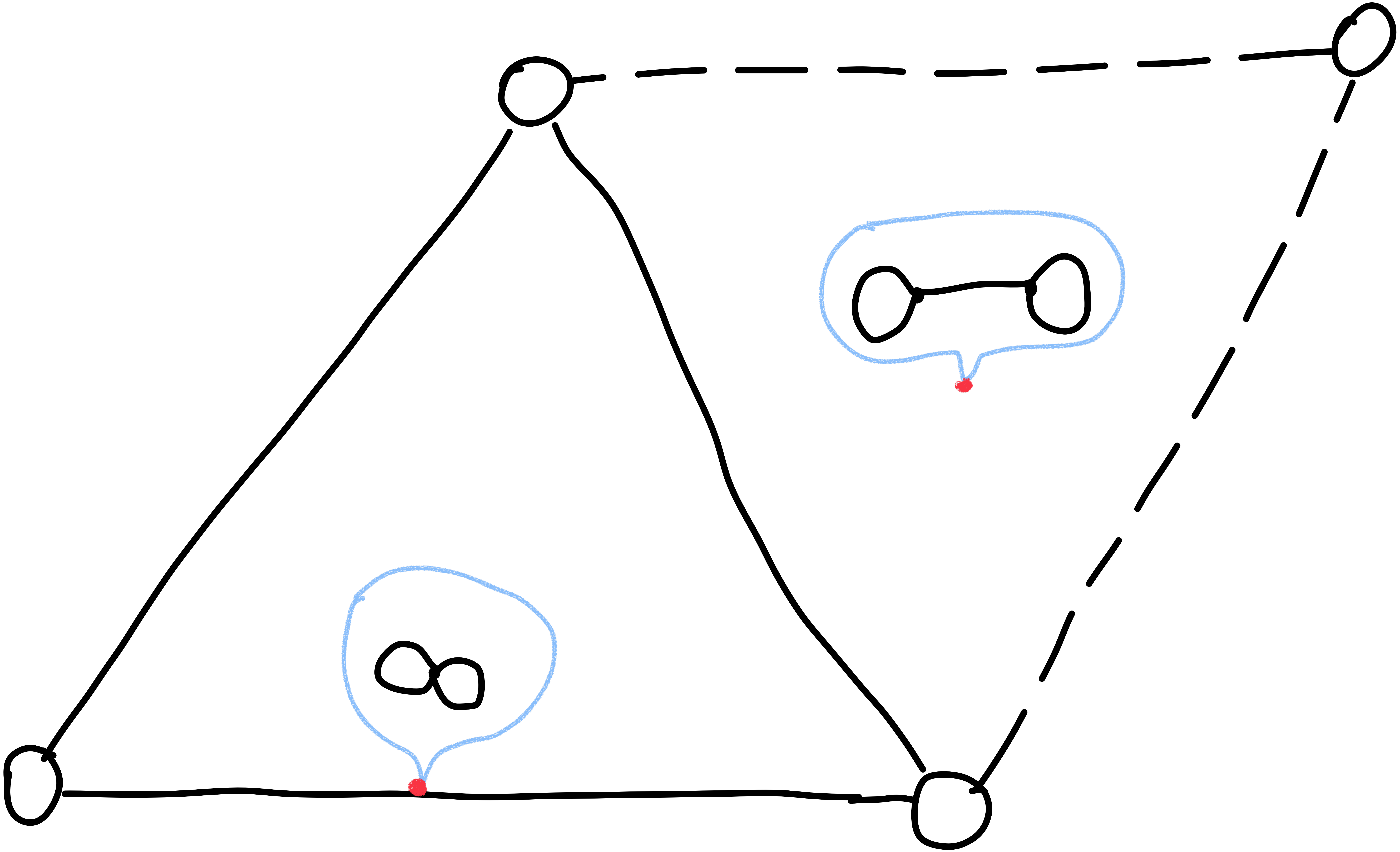
$\mathcal{M}_{2,0}^{\mathrm{tr}}$

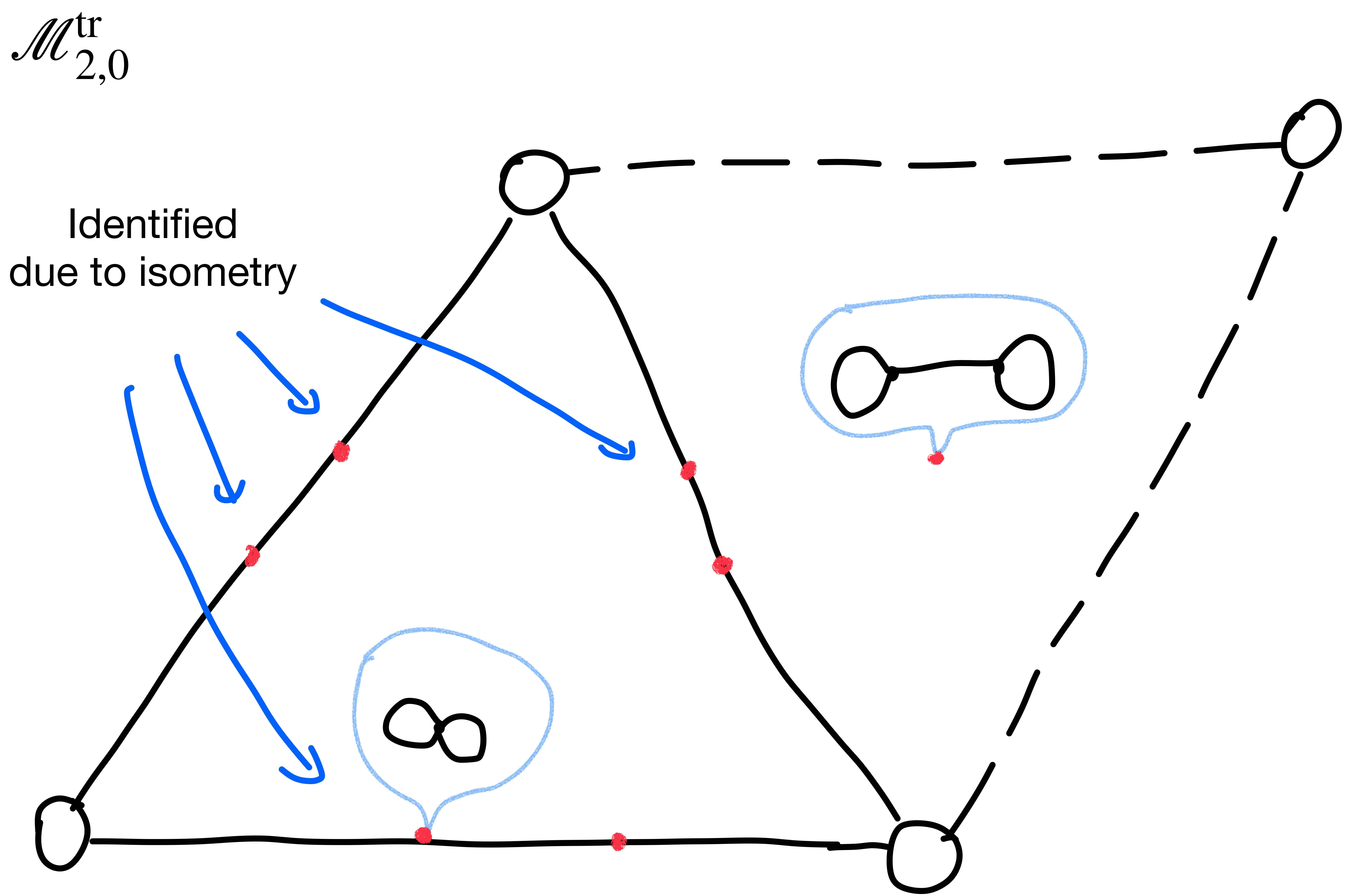


$\mathscr{M}^{\mathrm{tr}}_{2,0}$



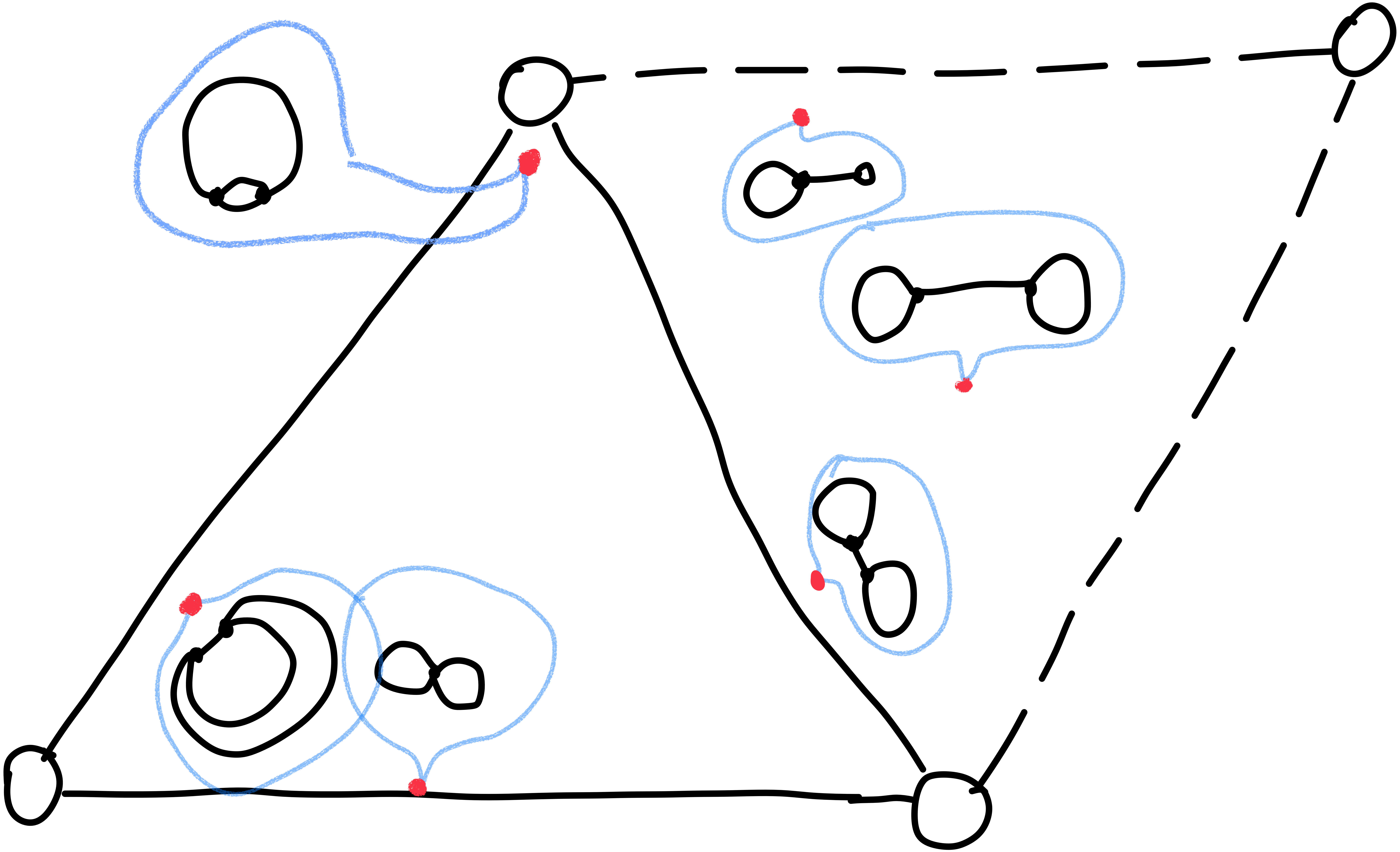
$\mathscr{M}^{\mathrm{tr}}_{2,0}$





$\mathscr{M}^{\mathrm{tr}}_{2,0}$
Identified
due to isometry

$\mathscr{M}^{\mathrm{tr}}_{2,0}$



$$\mathscr{A}_{L,n}(p_1, \ldots, p_n) = \sum_G \frac{1}{|\,\mathrm{Aut}G\,|} \int_{\mathbb{R}_{>0}^{E_G}} \frac{d\alpha_1 \ldots d\alpha_n}{\mathscr{P}_G(\boldsymbol{\alpha}, \boldsymbol{p})^{D/2}}$$

$$= \int_{\mathscr{M}_{L,n}^{\mathrm{tr}}} \mu_D(\boldsymbol{p})$$

Volume form on the moduli space.
(Takes form of Feynman integrand on each graph stratum)

$$\mathscr{A}_{L,n}(p_1, \ldots, p_n) = \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \mu_D(\boldsymbol{p})$$

Analogy to

Mirzakhani's volume recursion on moduli spaces of hyperbolic surfaces

$$V_{g,n}(L_1,\ldots,L_n) = \int_{\mathcal{M}_{g,n}(L_1,\ldots,L_n)} \omega_{WP}$$

Weil—Petersson form

Moduli space of hyperbolic surfaces of genus g with boundaries of length $L_1, \ldots, L_n$.

Theorem

Mirzakhani 2007 $V_{g,n}(L_1, \ldots, L_n)$ respects recursion equations (over all g, n).

$\Rightarrow$ The volumes $V_{g,n}(L_1, \ldots, L_n)$ can be computed easily.

$\Rightarrow$ Allows the study properties of **random surfaces** distributed as ω_{WP}.

Can we find a similar recursion for amplitudes?

$$\mathscr{A}_{L,n}(p_1, \ldots, p_n) = \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \mu_D(\boldsymbol{p})$$

For $L = 0$, yes $\to$ **Berens—Giele 1988**, **BCFW 2005, …**

For $L > 0$, no…

(Likely impossible by **Belkale—Brosnan 2000** and **Mnëv** universality **1985**)

Tropicalize…

$$\mathscr{A}_{L,n}(p_1, \ldots, p_n) = \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \mu_D(\boldsymbol{p})$$

Tropicalized amplitudes

$$\mathcal{A}^{\mathrm{tr}}_{L,n}(p_1, \ldots, p_n) = \int_{\mathcal{M}^{\mathrm{tr}}_{L,n}} \mu^{\mathrm{tr}}_{D}(\boldsymbol{p})$$

Tropicalized amplitudes

$$\mathscr{A}^{\mathrm{tr}}_{L,n}(p_1, \ldots, p_n) = \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \mu^{\mathrm{tr}}_D(\boldsymbol{p})$$

Theorem

MB 2025

$\mathscr{A}^{\mathrm{tr}}_{L,n}$ respects recursion equations (over all L, n).

$\Rightarrow$ The amplitudes $\mathscr{A}^{\mathrm{tr}}_{L,n}(p_1, \ldots, p_n)$ can be computed easily.

$\Rightarrow$ We can study properties of **random graphs** distributed as $\mu^{\mathrm{tr}}_D(\boldsymbol{p})$.

Tropicalization

Polynomial $$P(x_1, \ldots, x_n) = \sum_{\boldsymbol{k} \in M} a_{\boldsymbol{k}} \prod_{i=1}^{n} x_i^{k_i} \in \mathbb{R}_+[x_1, \ldots, x_n]$$

Definition
'Tropical approximation' $$P^{\mathrm{tr}}(x_1, \ldots, x_n) := \max_{\boldsymbol{k} \in M} \prod_{i=1}^{n} x_i^{k_i}$$

Rule: $\sum \to \max$ and $a_{\boldsymbol{k}} \to 1$.

'Typical' tropicalization is recovered in $\log$ coordinates.

Tropicalization

Polynomial $$P(x_1, \ldots, x_n) = \sum_{\boldsymbol{k} \in M} a_{\boldsymbol{k}} \prod_{i=1}^{n} x_i^{k_i} \in \mathbb{R}_+[x_1, \ldots, x_n]$$

Definition
'Tropical approximation'
$$P^{\mathrm{tr}}(x_1, \ldots, x_n) := \max_{\boldsymbol{k} \in M} \prod_{i=1}^{n} x_i^{k_i}$$

Lemma

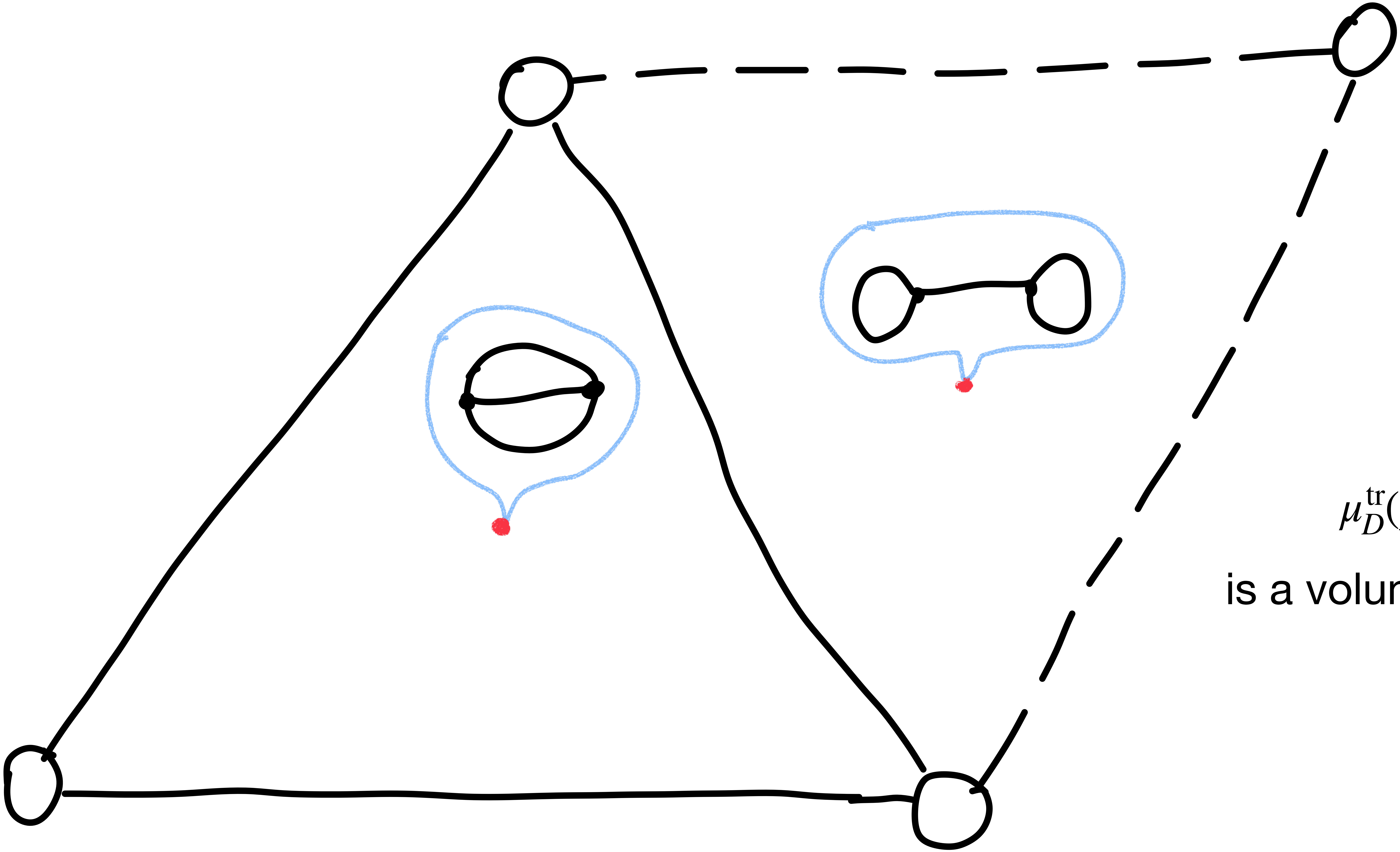
There are $C_1, C_2 > 0$, such that $C_1 \le \dfrac{P(x)}{P^{\mathrm{tr}}(x)} \le C_2$ for all $x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$.

Tropicalized amplitudes

$$\mathcal{A}^{\mathrm{tr}}_{L,n}(p_1, \ldots, p_n) = \int_{\mathcal{M}^{\mathrm{tr}}_{L,n}} \mu^{\mathrm{tr}}_D(\boldsymbol{p})$$

$$\mu_D(\boldsymbol{p})\,|_{\Delta_G} = \frac{d\alpha_1 \ldots d\alpha_n}{\mathcal{P}_G(\boldsymbol{\alpha}, \boldsymbol{p})^{D/2}} \quad \Rightarrow \quad \mu^{\mathrm{tr}}_D(\boldsymbol{p})\,|_{\Delta_G} = \frac{d\alpha_1 \ldots d\alpha_n}{\mathcal{P}^{\mathrm{tr}}_G(\boldsymbol{\alpha}, \boldsymbol{p})^{D/2}}$$

$\mathscr{M}^{\mathrm{tr}}_{2,0}$



$\mu^{\mathrm{tr}}_{D}(\boldsymbol{p})$

is a volume form

More physical:
Lagrangian formulation

Massive scalar quantum field theory

$$V[\Phi] = \sum_{k\geq 3} \lambda_k \Phi^k / k!$$

Action $$\mathcal{S}[\Phi, J] = \int_{\mathbb{R}^D} \mathrm{d}^D x \left(\frac{1}{2}\Phi(x)\left(\Box + m^2\right)\Phi(x) - V[\Phi](x) - J(x)\Phi(x) \right);$$

Massive scalar quantum field theory

$$V[\Phi] = \sum_{k \geq 3} \lambda_k \Phi^k / k!$$

Action

$$\mathcal{S}^{\xi}[\Phi, J] = \int_{\mathbb{R}^{D \cdot \xi}} \mathrm{d}^{D \cdot \xi} x \left(\frac{1}{2} \Phi(x) \left(\Box + m^2 \right)^{\xi} \Phi(x) - V[\Phi](x) - J(x) \Phi(x) \right),$$

Deformation: $D \to D \cdot \xi$ and $(\Box + m^2) \to (\Box + m^2)^{\xi}$

(Relative mass scalings of all operators remain constant.)

Massive scalar quantum field theory

$$V[\Phi] = \sum_{k\geq 3} \lambda_k \Phi^k / k!$$

Action

$$\mathcal{S}^\xi[\Phi, J] = \int_{\mathbb{R}^{D\cdot\xi}} \mathrm{d}^{D\cdot\xi}x \left(\frac{1}{2}\Phi(x)\left(\Box + m^2\right)^\xi \Phi(x) - V[\Phi](x) - J(x)\Phi(x)\right),$$

Partition function

$$Z^\xi[J] = \int \exp\left(-\mathcal{S}^\xi[\Phi, J]\right) \mathcal{D}[\Phi]$$

Massive scalar quantum field theory

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Partition function $$Z^\xi[J] = \int \exp\left(-\mathcal{S}^\xi[\Phi, J]\right) \mathcal{D}[\Phi]$$

Initial QFT

$$\xi = 1$$

$$\longrightarrow$$

Theorem MB 2025

$$\lim_{\xi\to 0^+} Z^\xi = Z^{\mathrm{tr}}$$

Tropicalized QFT is a **deformation limit** of the initial QFT.

Effective action:

$$\Gamma^{\mathrm{tr}}(\hbar, \varphi) = \sum_{L,n} \mathscr{A}^{\mathrm{tr}}_{L,n} \hbar^L \varphi^n$$

Recursive solution $\leftrightarrow$ exact solvability

Theorem **(MB 2025)**

Perturbatively, the tropicalized QFT is completely fixed by the nonlinear PDE

$$\mathscr{P}_D \Gamma^{\mathrm{tr}} = \left(1 - \frac{\partial^2 \Gamma^{\mathrm{tr}}}{\partial \varphi^2}\right)^{-1} - 1, \qquad \text{where}$$

Recursive solution ↔ exact solvability

Theorem (MB 2025)

Perturbatively, the tropicalized QFT is completely fixed by the nonlinear PDE

$$\mathscr{P}_D \Gamma^{\mathrm{tr}} = \left(1 - \frac{\partial^2 \Gamma^{\mathrm{tr}}}{\partial \varphi^2}\right)^{-1} - 1, \qquad \text{where}$$

Γ^{tr} is the tropicalized QFT's **effective action**, and

$$\mathcal{P}_D = \left(-D - \left(1 - \frac{D}{2}\right) \varphi \frac{\partial}{\partial \varphi} + \sum_{k \geq 3} \left(k - D\left(\frac{k}{2} - 1\right)\right) \lambda_k \frac{\partial}{\partial \lambda_k} \right)$$

(+ trivial boundary conditions.)

Proof of the solvability/recursion

Exploits a recursive, combinatorial relation between tropicalized Feynman integrals

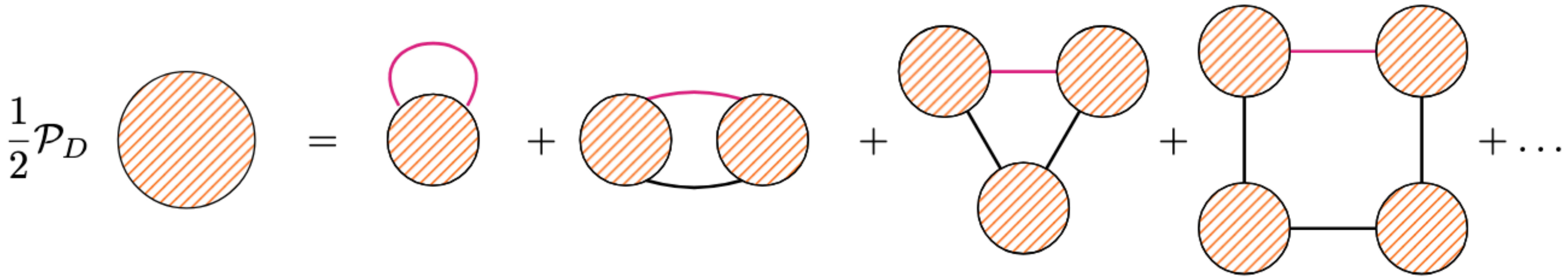


Figure 3: Illustration of the tropical loop equation. Each orange blob stands for a 1PI graph. The right-hand graphs have one purple pointed edge. Cutting these edges yields beaded graphs.

$$
\begin{aligned}
\Gamma^{\mathrm{tr}} = & -\frac{\varphi\lambda_3}{D-2} + \frac{\lambda_4}{2(D-2)^2} + \frac{\lambda_3^2}{(D-3)(D-4)} + \varphi\lambda_3^3\frac{5D-24}{(D-4)^2(D-6)} + \frac{\varphi\lambda_5}{2(D-2)^2} \\
& +\varphi\lambda_3\lambda_4\frac{2(2D-5)}{(D-2)(D-3)(D-4)} - \frac{\varphi^2\lambda_3^2}{D-4} - \frac{\varphi^2\lambda_4}{2(D-2)} - 2\frac{\lambda_3\lambda_5}{(D-2)(D-3)(D-4)} \\
& -\frac{2}{3}\lambda_3^4\frac{-2400+1412D-272D^2+17D^3}{(D-4)^3(D-5)(D-6)(D-8)} - \frac{\lambda_6}{6(D-2)^3} - \lambda_4^2\frac{32-25D+5D^2}{(D-2)^2(D-3)(D-4)(3D-8)} \\
& -\lambda_3^2\lambda_4\frac{480-290D+39D^2}{(D-2)(D-4)^2(D-6)(3D-10)} + \dots
\end{aligned}
$$

Tropicalized amplitudes

$$\mathscr{A}^{\mathrm{tr}}_{L,n}(p_1, \ldots, p_n) = \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \mu^{\mathrm{tr}}_D(\boldsymbol{p})$$

Theorem

MB 2025

$\mathscr{A}^{\mathrm{tr}}_{L,n}$ respects recursion equations (over all L, n).

$\Rightarrow$ The amplitudes $\mathscr{A}^{\mathrm{tr}}_{L,n}(p_1, \ldots, p_n)$ can be computed easily.

$\Rightarrow$ We can study properties of **random graphs** distributed as $\mu^{\mathrm{tr}}_D(\boldsymbol{p})$.

Going back to the non-tropicalized QFT

$$
\begin{aligned}
\mathscr{A}_{L,n}(p_1,\ldots,p_n) &= \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \mu_D(\boldsymbol{p}) \\
&= \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \left(\frac{\mathscr{P}^{\mathrm{tr}}_G(\boldsymbol{\alpha},\boldsymbol{p})}{\mathscr{P}_G(\boldsymbol{\alpha},\boldsymbol{p})} \right)^{D/2} \mu^{\mathrm{tr}}_D(\boldsymbol{p}) \\
&= \mathscr{A}^{\mathrm{tr}}_{L,n} \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \left(\frac{\mathscr{P}^{\mathrm{tr}}_G(\boldsymbol{\alpha},\boldsymbol{p})}{\mathscr{P}_G(\boldsymbol{\alpha},\boldsymbol{p})} \right)^{D/2} \underset{\uparrow}{\widehat{\mu}^{\mathrm{tr}}_D(\boldsymbol{p})}
\end{aligned}
$$

Probability density

$$\mathscr{A}_{L,n}(p_1,\ldots,p_n) = \mathscr{A}^{\mathrm{tr}}_{L,n} \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \left(\frac{\mathscr{P}^{\mathrm{tr}}_G(\boldsymbol{\alpha},\boldsymbol{p})}{\mathscr{P}_G(\boldsymbol{\alpha},\boldsymbol{p})} \right)^{D/2} \widehat{\mu}^{\mathrm{tr}}_D(\boldsymbol{p})$$

Key: this term is bounded.

Numerical evaluation of $\mathscr{A}_{L,n}(p_1,\ldots,p_n)$ $\Longleftrightarrow$ Sampling algorithms for $\int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \widehat{\mu}^{\mathrm{tr}}_D(\boldsymbol{p}) = 1.$

Monte Carlo

Theorem

MB 2025

There is a **global sampling** algorithm for the measure $\widehat{\mu}^{\mathrm{tr}}_{D}(\boldsymbol{p})$ on $\mathscr{M}^{\mathrm{tr}}_{L,n}$.

Theorem

MB 2025

There is a **global sampling** algorithm for the measure $\widehat{\mu}_D^{\mathrm{tr}}(\boldsymbol{p})$ on $\mathscr{M}_{L,n}^{\mathrm{tr}}$.

It's runtime and memory requirements scale **polynomially.**

Theorem

MB 2025

There is a **global sampling** algorithm for the measure $\widehat{\mu}_D^{\mathrm{tr}}(\boldsymbol{p})$ on $\mathscr{M}_{L,n}^{\mathrm{tr}}$.

It's runtime and memory requirements scale **polynomially.**

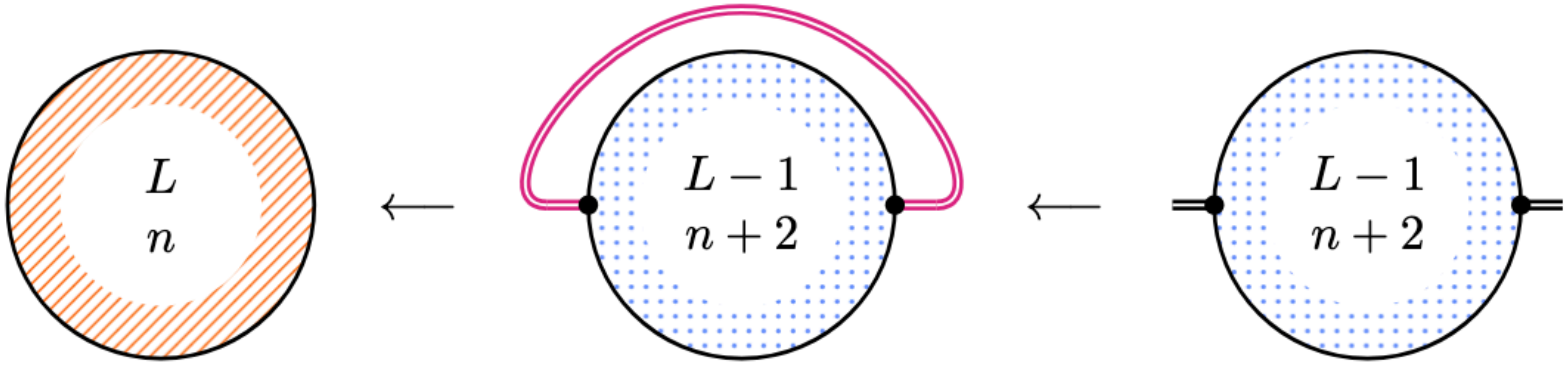
Proof: based on exact solution of tropicalized QFT.

$\Rightarrow \mathscr{A}_{L,n}(p_1, \ldots, p_n)$ can be (in principle) be estimated in **polynomial-time** in L and n.
(Accuracy still scales worse.)

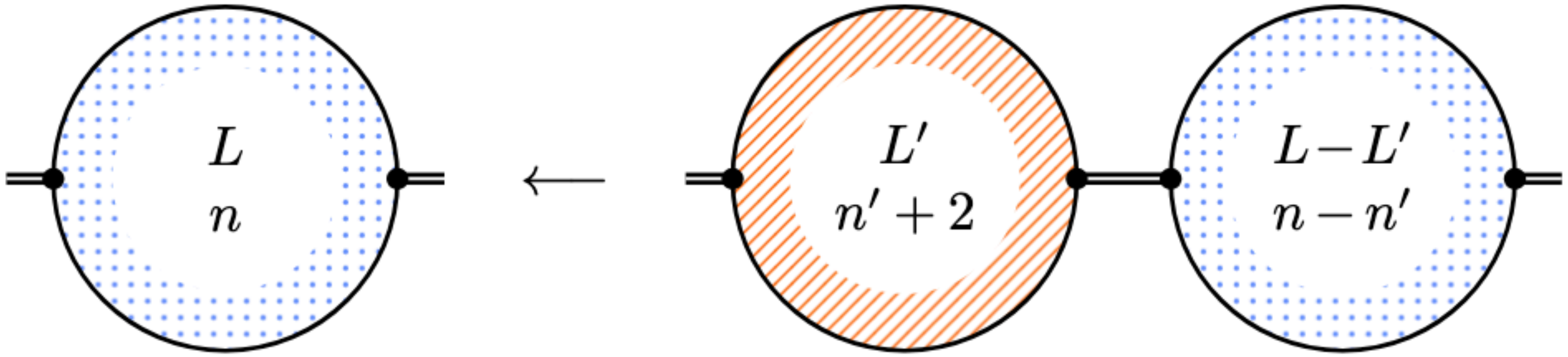
Fastest algorithms for Feynman integration run in exponential time.

Fastest algorithms for Feynman integration run in exponential time.

$\Rightarrow$ Evaluation of individual Feynman integrals quickly becomes slower than evaluating $\mathscr{A}_{L,n}(p_1, \ldots, p_n)$.



(a) Illustration of the $L > 0$ case of Algorithm 18.



(b) Illustration of the (L', n') case in Algorithm 19.

The algorithm recursively produces metric graphs with the correct probability distribution. Most graphs will never be generated.

Example computations

Massive ϕ^3 theory in $D = 3$, with $m^2 = 1$.

Relations to **Lee—Yang 1952** phenomenon

Known (numerically) up to $L \leq 7$

(**Sberveglieria, Spada 2024**)

L	samples	$\widetilde{\Gamma}_{\bar{L},3}(0,0,0)$	time/h
1	$1 \cdot 10^{11}$	$4.431109 \cdot 10^{-1} \pm 1.1 \cdot 10^{-6}$	2
2	$1 \cdot 10^{11}$	$1.047191 \cdot 10^{0} \pm 5.9 \cdot 10^{-6}$	3
3	$1 \cdot 10^{11}$	$2.902190 \cdot 10^{0} \pm 3.6 \cdot 10^{-5}$	3
4	$1 \cdot 10^{11}$	$8.877142 \cdot 10^{0} \pm 2.7 \cdot 10^{-4}$	4
5	$1 \cdot 10^{11}$	$2.920635 \cdot 10^{1} \pm 2.4 \cdot 10^{-3}$	6
6	$1 \cdot 10^{11}$	$1.019640 \cdot 10^{2} \pm 2.4 \cdot 10^{-2}$	6
7	$1 \cdot 10^{11}$	$3.748502 \cdot 10^{2} \pm 3.3 \cdot 10^{-1}$	7
8	$1 \cdot 10^{11}$	$1.440633 \cdot 10^{3} \pm 2.1 \cdot 10^{0}$	8
9	$1 \cdot 10^{11}$	$5.787627 \cdot 10^{3} \pm 2.2 \cdot 10^{1}$	9
10	$1 \cdot 10^{11}$	$2.399101 \cdot 10^{4} \pm 1.4 \cdot 10^{2}$	7
11	$1 \cdot 10^{11}$	$1.074911 \cdot 10^{5} \pm 2.6 \cdot 10^{3}$	12
12	$1 \cdot 10^{11}$	$4.760706 \cdot 10^{5} \pm 1.2 \cdot 10^{4}$	13
13	$1 \cdot 10^{11}$	$2.235488 \cdot 10^{6} \pm 1.0 \cdot 10^{5}$	15
14	$1 \cdot 10^{11}$	$1.000354 \cdot 10^{7} \pm 3.3 \cdot 10^{5}$	16
15	$1 \cdot 10^{11}$	$5.464614 \cdot 10^{7} \pm 4.0 \cdot 10^{6}$	16
16	$1 \cdot 10^{11}$	$2.859931 \cdot 10^{8} \pm 3.4 \cdot 10^{7}$	17
17	$1 \cdot 10^{11}$	$1.156947 \cdot 10^{9} \pm 3.6 \cdot 10^{7}$	20
18	$1 \cdot 10^{11}$	$8.861573 \cdot 10^{9} \pm 1.6 \cdot 10^{9}$	20
19	$1 \cdot 10^{11}$	$7.159013 \cdot 10^{10} \pm 3.6 \cdot 10^{10}$	23
20	$1 \cdot 10^{11}$	$2.776484 \cdot 10^{11} \pm 5.2 \cdot 10^{10}$	24

Table 1: 3-point function computation in massive ϕ^3 theory in $D = 3$.

Primitive β function in 4-dim ϕ^4 theory

(Conjectured to be equal to full MS β function for large L.)

Relation to the **Ising model**

Known before (analytically) for $L \leq 7$ **Schnetz 2022.**

With tropical sampling (numerically) up to $L \leq 18$ **Balduf 2023**.

L	samples	N_{Prim}	$\beta^{\text{prim}}_{L+1}$	$\beta H^{\text{prim}}_{L+1}$	time/h
3	$1.10 \cdot 10^{10}$	$1.87 \cdot 10^{9}$	$1.442497 \cdot 10^{1} \pm 3.0 \cdot 10^{-4}$	$1.679980 \cdot 10^{2} \pm 3.5 \cdot 10^{-3}$	0
4	$1.10 \cdot 10^{10}$	$1.31 \cdot 10^{9}$	$1.244281 \cdot 10^{2} \pm 3.5 \cdot 10^{-3}$	$3.432005 \cdot 10^{3} \pm 8.9 \cdot 10^{-2}$	1
5	$1.10 \cdot 10^{10}$	$1.28 \cdot 10^{9}$	$1.698163 \cdot 10^{3} \pm 5.5 \cdot 10^{-2}$	$1.135437 \cdot 10^{5} \pm 3.0 \cdot 10^{0}$	1
6	$1.10 \cdot 10^{10}$	$1.18 \cdot 10^{9}$	$2.412932 \cdot 10^{4} \pm 9.1 \cdot 10^{-1}$	$3.958005 \cdot 10^{6} \pm 1.1 \cdot 10^{2}$	1
7	$1.10 \cdot 10^{10}$	$1.10 \cdot 10^{9}$	$3.709545 \cdot 10^{5} \pm 1.6 \cdot 10^{1}$	$1.509371 \cdot 10^{8} \pm 4.3 \cdot 10^{3}$	1
8	$1.10 \cdot 10^{10}$	$1.04 \cdot 10^{9}$	$6.062108 \cdot 10^{6} \pm 3.1 \cdot 10^{2}$	$6.179273 \cdot 10^{9} \pm 1.8 \cdot 10^{5}$	2
9	$1.10 \cdot 10^{10}$	$9.80 \cdot 10^{8}$	$1.045110 \cdot 10^{8} \pm 6.2 \cdot 10^{3}$	$2.692812 \cdot 10^{11} \pm 8.2 \cdot 10^{6}$	2
10	$1.10 \cdot 10^{10}$	$9.33 \cdot 10^{8}$	$1.889201 \cdot 10^{9} \pm 1.3 \cdot 10^{5}$	$1.241497 \cdot 10^{13} \pm 3.9 \cdot 10^{8}$	3
11	$1.10 \cdot 10^{10}$	$8.96 \cdot 10^{8}$	$3.566923 \cdot 10^{10} \pm 2.8 \cdot 10^{6}$	$6.026765 \cdot 10^{14} \pm 1.9 \cdot 10^{10}$	4
12	$1.10 \cdot 10^{10}$	$8.66 \cdot 10^{8}$	$7.012027 \cdot 10^{11} \pm 6.4 \cdot 10^{7}$	$3.071324 \cdot 10^{16} \pm 1.0 \cdot 10^{12}$	5
13	$1.10 \cdot 10^{10}$	$8.44 \cdot 10^{8}$	$1.431902 \cdot 10^{13} \pm 1.5 \cdot 10^{9}$	$1.638982 \cdot 10^{18} \pm 5.4 \cdot 10^{13}$	6
14	$1.10 \cdot 10^{10}$	$8.28 \cdot 10^{8}$	$3.032472 \cdot 10^{14} \pm 3.6 \cdot 10^{10}$	$9.142727 \cdot 10^{19} \pm 3.1 \cdot 10^{15}$	7
15	$3.11 \cdot 10^{11}$	$2.31 \cdot 10^{10}$	$6.655768 \cdot 10^{15} \pm 2.4 \cdot 10^{10}$	$5.323570 \cdot 10^{21} \pm 3.4 \cdot 10^{16}$	249
16	$1.10 \cdot 10^{10}$	$8.10 \cdot 10^{8}$	$1.512467 \cdot 10^{17} \pm 2.4 \cdot 10^{13}$	$3.231993 \cdot 10^{23} \pm 1.1 \cdot 10^{19}$	11
17	$1.10 \cdot 10^{10}$	$8.08 \cdot 10^{8}$	$3.552250 \cdot 10^{18} \pm 6.3 \cdot 10^{14}$	$2.044094 \cdot 10^{25} \pm 6.9 \cdot 10^{20}$	12
18	$1.10 \cdot 10^{10}$	$8.09 \cdot 10^{8}$	$8.632116 \cdot 10^{19} \pm 1.8 \cdot 10^{16}$	$1.345581 \cdot 10^{27} \pm 4.6 \cdot 10^{22}$	15
19	$1.10 \cdot 10^{10}$	$8.12 \cdot 10^{8}$	$2.167796 \cdot 10^{21} \pm 4.9 \cdot 10^{17}$	$9.211519 \cdot 10^{28} \pm 3.1 \cdot 10^{24}$	19
20	$3.00 \cdot 10^{11}$	$2.23 \cdot 10^{10}$	$5.624473 \cdot 10^{22} \pm 4.0 \cdot 10^{17}$	$6.551806 \cdot 10^{30} \pm 4.2 \cdot 10^{25}$	621
25	$8.30 \cdot 10^{10}$	$6.50 \cdot 10^{9}$	$1.066295 \cdot 10^{30} \pm 2.9 \cdot 10^{25}$	$2.060052 \cdot 10^{40} \pm 2.5 \cdot 10^{35}$	824
30	$1.00 \cdot 10^{11}$	$8.26 \cdot 10^{9}$	$4.290822 \cdot 10^{37} \pm 1.8 \cdot 10^{33}$	$1.486361 \cdot 10^{50} \pm 1.6 \cdot 10^{45}$	2032
40	$1.00 \cdot 10^{10}$	$8.86 \cdot 10^{8}$	$4.946806 \cdot 10^{53} \pm 1.6 \cdot 10^{50}$	$6.283492 \cdot 10^{70} \pm 2.0 \cdot 10^{66}$	638
50	$1.00 \cdot 10^{10}$	$9.26 \cdot 10^{8}$	$5.054951 \cdot 10^{70} \pm 3.8 \cdot 10^{67}$	$2.625921 \cdot 10^{92} \pm 8.2 \cdot 10^{87}$	725

Table 2: Primitive β function computation in ϕ^4 theory.

Conclusion and open questions

- Tropicalized QFT is a deformed, **exactly solvable** version of (scalar) QFT.
- Gives **volume recursions** on the moduli space of graphs/tropical curves.
- Gives a **polynomial-time** algorithm for estimation of scattering amplitudes.
- Many possibilities for extensions and generalizations (e.g. to **non-scalar QFTs**).
- **New numerical problems** need to be tamed: even though sampling works in polynomial time, perturbative coefficients take exponential time in the **relative** target accuracy.
- What is the **non-perturbative** nature of tropicalized QFT?
- What is the **large loop order** behavior (tropicalized and non-tropicalized)? (some answers…)
- **Renormalization** of tropicalized QFT? **Renormalized** sampling?
- Does the solution of tropicalized QFT constrain the original? **Bootstrap**?
- Statistical study of **arithmetic** properties of Feynman integrals: E.g. **average** symbol?