Systematic integration of higher loop gauge theory amplitudes in position space

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September 12, RADCOR 2019

joint work with Oliver Schnetz

Why position space?

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Advantages

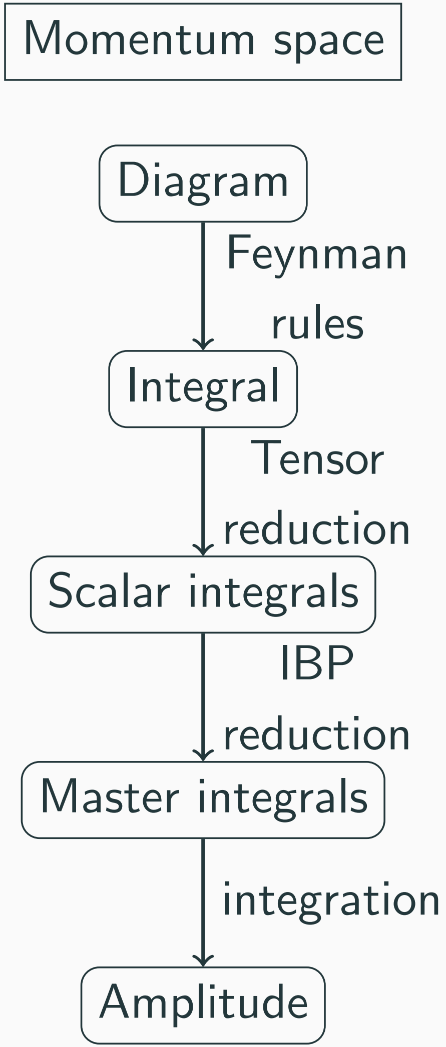
- Simpler Feynman rules
- No IBP reduction necessary
- Conceptually interesting viewpoint

Caveats

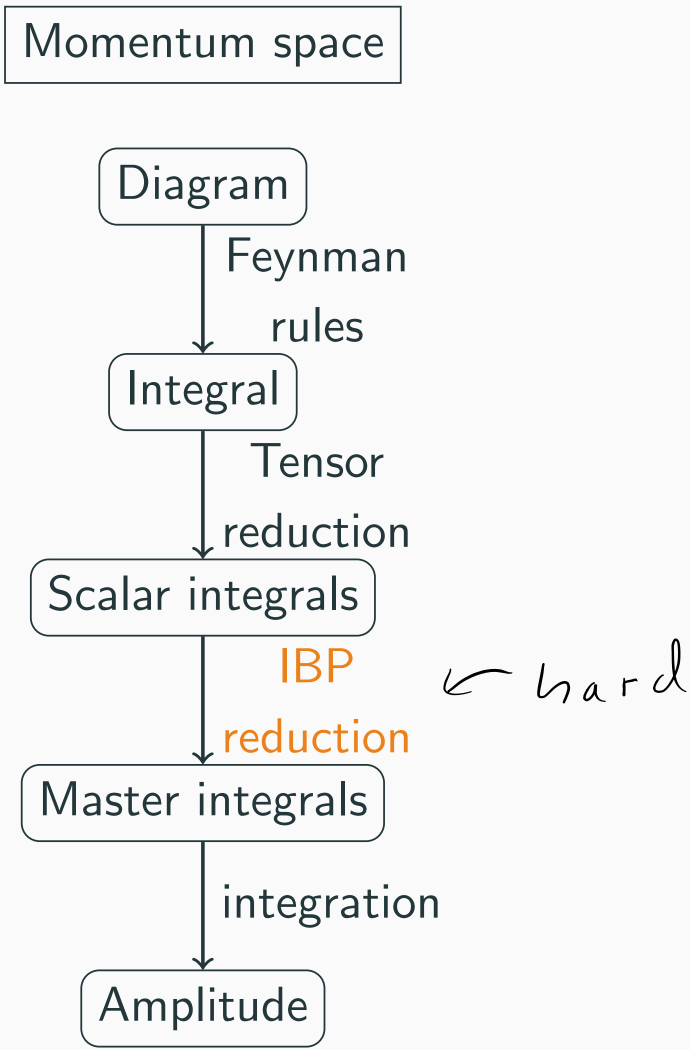
- Limited applications: only renormalization quantities so far
- New technology needed

Proof of concept:
7-loop β-function in ϕ^4 calculated in 2016 by Oliver Schnetz using graphical functions.

Loop integral workflow



Loop integral workflow



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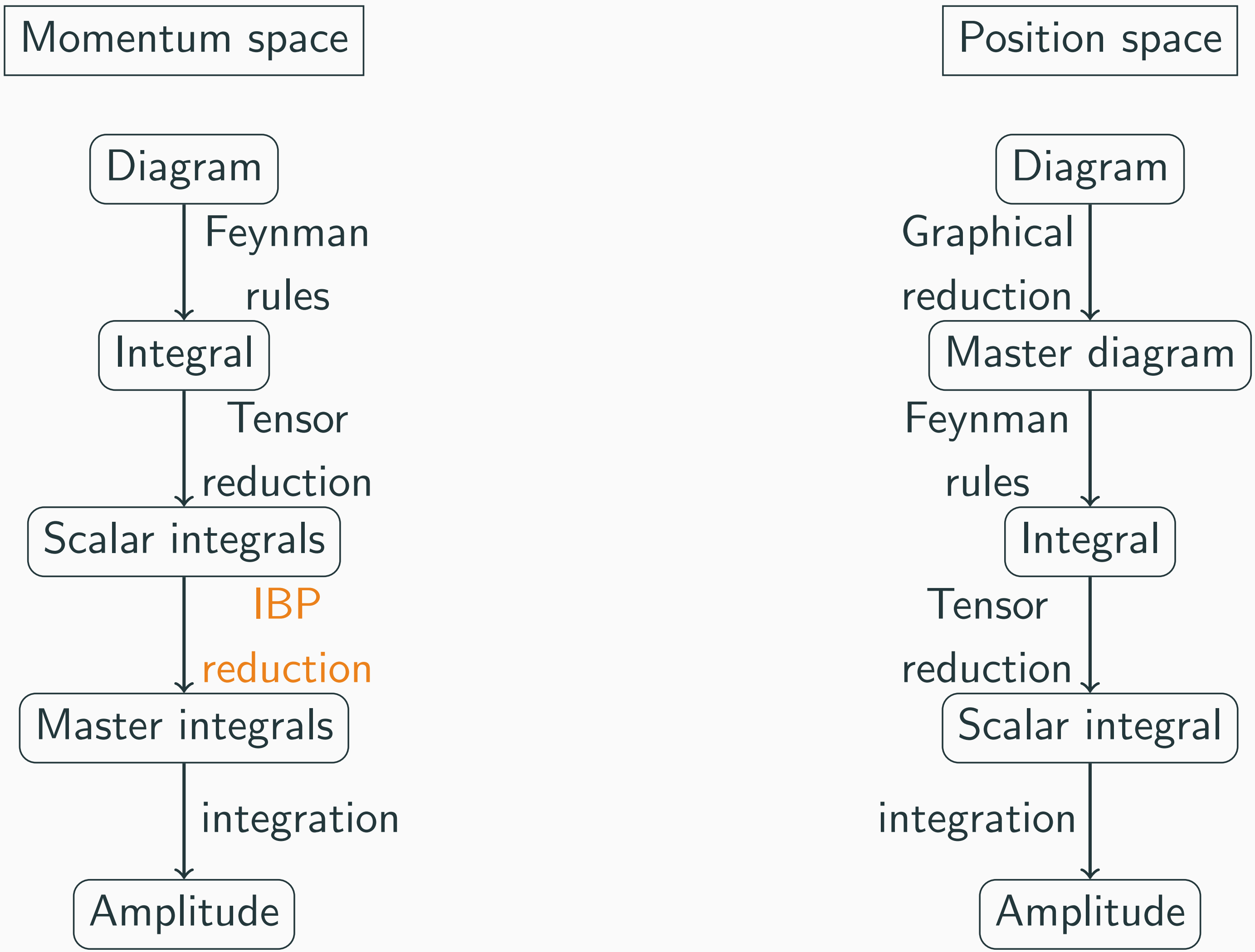

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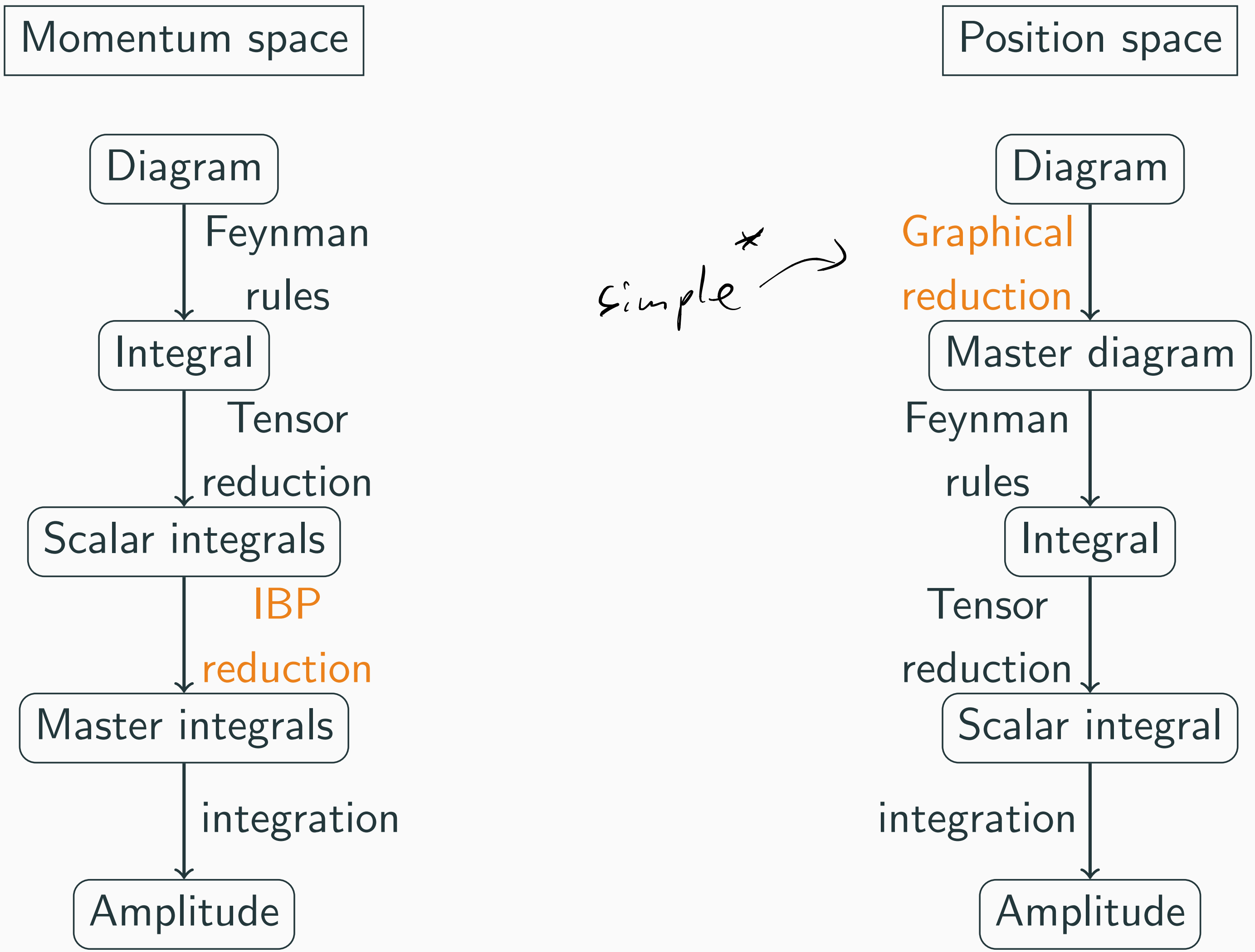


Feynman integral in momentum space

$$\widetilde{G}(p_1, \ldots, p_n) = \left(\prod_{e \in E} \int d^D k_e \widetilde{\Delta}(k_e) \right) \underbrace{\left(\prod_{v \in V_{\text{int}}} \delta^{(D)} \left(\sum_{e \ni v} k_e \right) \right)}$$

→ Lower dimensional integral

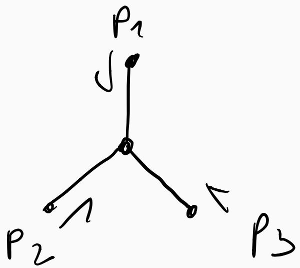
Feynman integral in position space

$$G(x_1, \ldots, x_n) = \left(\prod_{v \in V_{\text{int}}} \int d^D x_v \right) \underbrace{\left(\prod_{\{a,b\} \in E} \Delta(x_a - x_b) \right)}$$

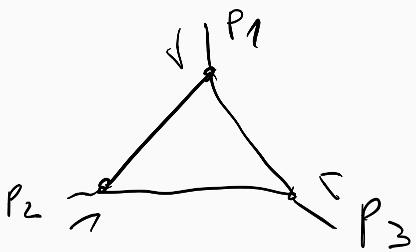
→ Better factorization properties

Examples

Momentum space

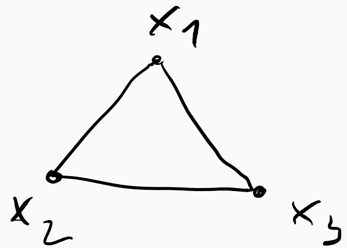


$\widetilde{\Delta}(p_{12})\widetilde{\Delta}(p_{23})\widetilde{\Delta}(p_{31})$

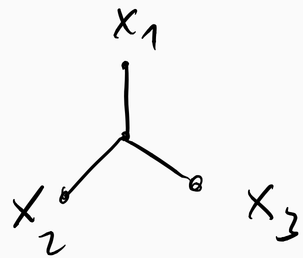


$$\frac{\mathrm{Di}(z,\bar{z})}{\sqrt{-\lambda(p_{12}^2,p_{23}^2,p_{31}^2)}}$$

Position space



$\Delta(x_{12})\Delta(x_{23})\Delta(x_{31})$



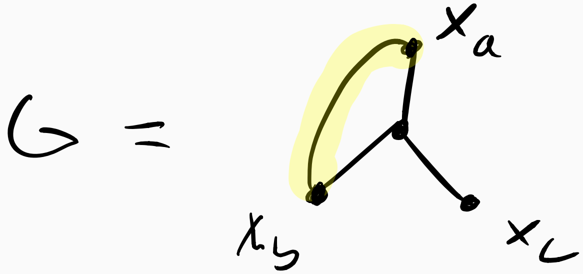
$$\frac{\mathrm{Di}(z,\bar{z})}{\sqrt{-\lambda(x_{12}^2,x_{23}^2,x_{31}^2)}}$$

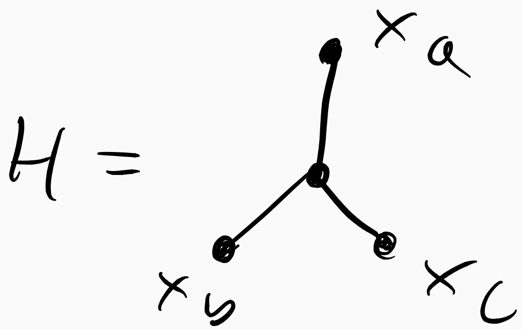
Graphical reductions

Graphical reduction rules

1. rule: propogators between external vertices

$$G(x_a, x_b, x_c) = \int d^D y \Delta(x_a - y)\Delta(x_b - y)\Delta(x_c - y)\Delta(x_a - x_b)$$
$$= \Delta(x_a - x_b) H(x_a, x_b, x_c)$$

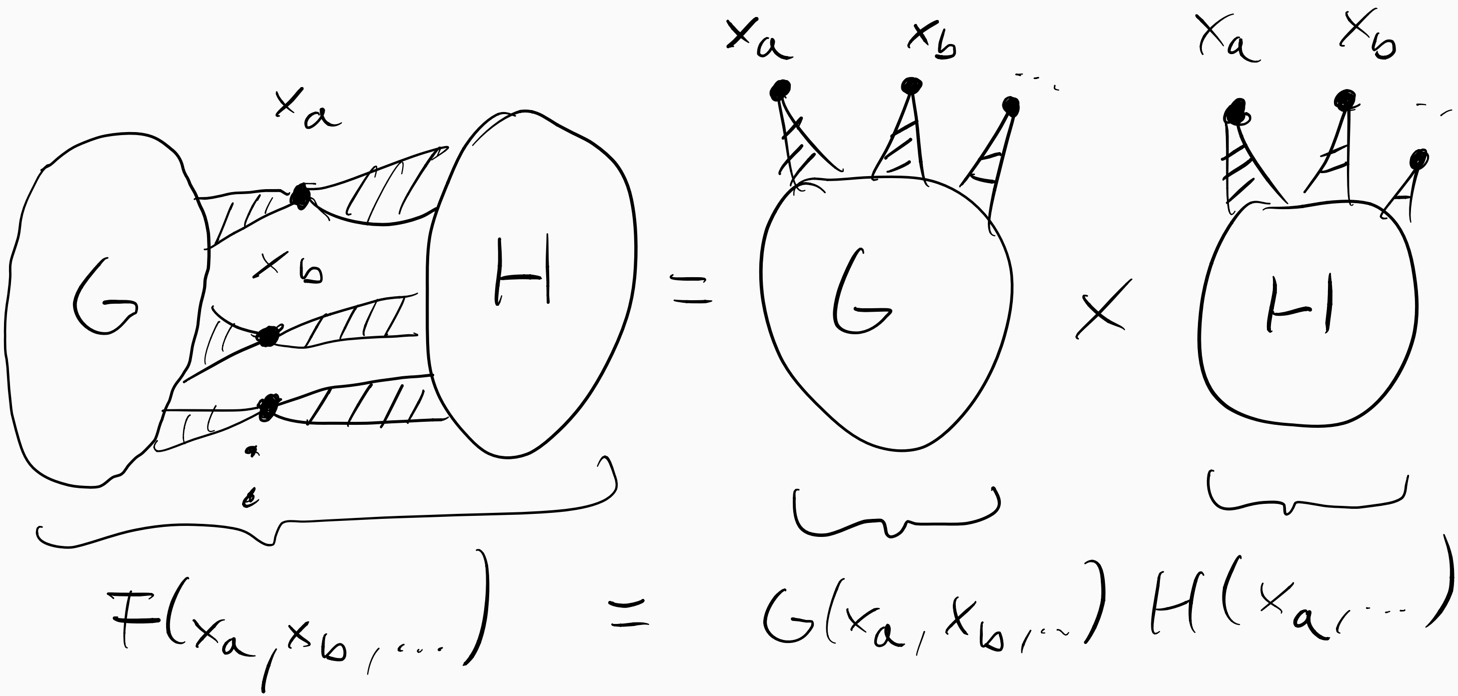




⇒ edges between external vertices factorize.

Graphical reduction rules

2. rule: split graph



⇒ factorizes if split along external vertices.

Intermezzo: amputating a propagator

Recall the definition of the propagator, Δ, as *Green's function for the free field equation*

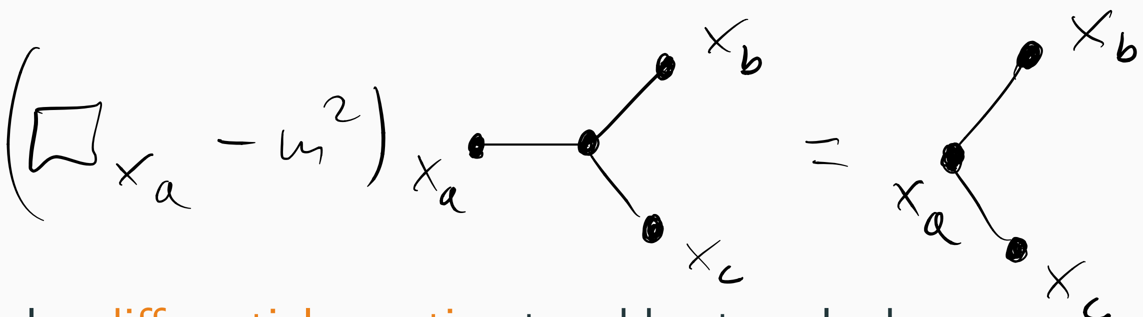
$$(\Box_x - m^2)\Delta(x - y) = \delta^{(D)}(x - y)$$

We can use this equation to *amputate* free external edges.

Graphical reduction rules

3. rule: amputating an external edge

$$(\Box_{x_a} - m^2)G(x_a, x_b, x_c) = \int d^D y (\Box_{x_a} - m^2)\Delta(x_a - y)\Delta(x_b - y)\Delta(x_c - y)$$
$$= \int d^D y \delta(x_a - y)\Delta(x_b - y)\Delta(x_c - y)$$
$$= \Delta(x_b - x_a)\Delta(x_c - x_a) = H(x_a, x_b, x_c)$$



$\Rightarrow$ solve differential equation to add external edge.

For rule 3, a differential equation needs to be solved:

$(\Box_{x_a} - m^2)G$  $(x_a, \ldots) = G$  $(x_a, \ldots)$

Can be solved systematically if (Schnetz 2013)

- particles are massless, $m = 0$,
- only 3-point functions are considered
- in $D = 4 - \epsilon$ Euklidean space.

3-point configuration space is 2-dimensional $\Rightarrow$

Use complex paramater z such that

$$z\,\bar{z} = \frac{x_{ac}^2}{x_{ab}^2} \quad \text{and} \quad (1-z)(1-\bar{z}) = \frac{x_{bc}^2}{x_{ab}^2}$$

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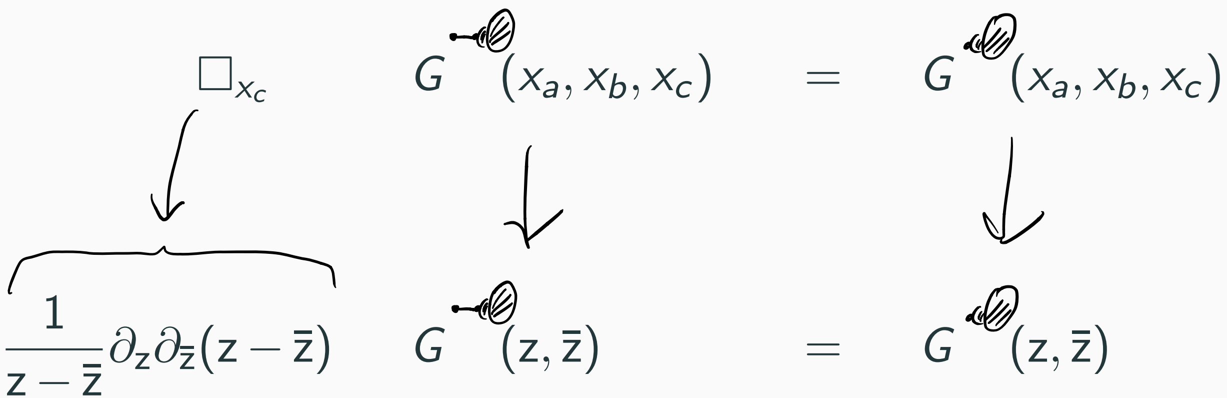

The ∂_z and $\partial_{\bar{z}}$ operators can be inverted in the function space of generalized single-valued hyperlogarithms (Chavez, Duhr 2012, Schnetz 2014, Schnetz 2017).

Graphical functions

- Rules 1,2,3 are part of a larger framework: graphical functions (Schnetz 2013).
- Graphical functions can also be applied in a broader context, e.g. to conformal amplitudes (Basso, Dixon 2017).
- Calculation within this framework are extremely efficient, due to the rapid reductions and small numbers of irreducible *master diagrams*.

Graphical functions for gauge theory

Beyond scalar

Only change: adding an edge

For instance, for abelian gauge theory:

$$\Box_x \to \not{\partial} \text{ and } \eta^{\mu\nu}\Box_x$$

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$$\Box_x \to \not\partial \text{ and } \eta^{\mu\nu}\Box_x$$

The differential equation for appending an edge,

$\Box_{x_a} G$$(x_a, \ldots) = G$$(x_a, \ldots)$

becomes a system of differential equations

$\not\partial_{x_a} G$$(x_a, \ldots) = G$$(x_a, \ldots)$

Paramatrizing non-scalar graphical functions

$$\not{\partial}_{x_c} \qquad G$$  $$(x_a, x_b, x_c) \quad = \quad G$$  $$(x_a, x_b, x_c)$$

Paramatrizing non-scalar graphical functions

$$\not{\partial}_{x_c} \, G(x_a, x_b, x_c) = G(x_a, x_b, x_c)$$

$$\left(\not{\lambda}\, \partial_z + \not{\bar\lambda}\, \partial_{\bar z} - \frac{P^{\mu\nu}}{z - \bar z} \left(\partial^\nu_{\not\lambda} - \partial^\nu_{\not{\bar\lambda}} \right) \right) G(z, \bar z, \lambda, \bar\lambda) = G(z, \bar z, \lambda, \bar\lambda)$$

Using light-cone-like parametrization $z, \bar z, \lambda^\mu, \bar\lambda^\mu$ such that

$$z\,\bar z = \frac{x_{ac}^2}{x_{ab}^2} \quad \text{and} \quad (1-z)(1-\bar z) = \frac{x_{bc}^2}{x_{ab}^2}$$

$$x^\mu_{ab} = \lambda^\mu + \bar\lambda^\mu \qquad x^\mu_{ac} = z\,\lambda^\mu + \bar z\,\bar\lambda^\mu \qquad x^\mu_{bc} = (1-z)\,\lambda^\mu + (1-\bar z)\,\bar\lambda^\mu$$

$$\lambda^\mu \lambda_\mu = \bar\lambda^\mu \bar\lambda_\mu = 0$$

Actual inversion becomes more complicated: $D \neq 4$ dimensional Laplacian has to be inverted.

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- Work in progress: extension to gauge theory.
- Intermediate step finished: extension to arbitrary even D.

Example of a master diagram, which is irreducible w.r.t. rules 1–3: