Tropical quantum field theory

Universe+

arXiv:2508.14263

March 25, 2026

Michael Borinsky — Perimeter Institute

Motivation

Quantum field theory

Slogan

Quantum Field Theory is the best theory

One of theoretical physics most precise tools

PHYSICAL REVIEW LETTERS **130,** 071801 (2023)

Editors' Suggestion **Featured in Physics**

Measurement of the Electron Magnetic Moment

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The electron magnetic moment, $-\mu/\mu_B = g/2 = 1.001\,159\,652\,180\,59\,(13)$ [0.13 ppt], is determined 2.2 times more accurately than the value that stood for fourteen years. The most precisely determined property of an elementary particle tests the most precise prediction of the standard model (SM) to 1 part in 10^{12}. The test would improve an order of magnitude if the uncertainty from discrepant measurements of the fine structure constant α is eliminated since the SM prediction is a function of α. The new measurement and SM theory together predict $\alpha^{-1} = 137.035\,999\,166\,(15)$ [0.11 ppb] with an uncertainty 10 times smaller than the current disagreement between measured α values.

DOI: 10.1103/PhysRevLett.130.071801

… theoretical computations up to
12 digits come from
perturbative quantum field theory

$$\mathscr{A}_{L,n}(p_1, \ldots, p_n) = \sum_{G} \frac{1}{|\mathrm{Aut}(G)|} I_G$$

QFT observable/
amplitude

Sum over graphs with
L loops and n legs

Symmetry
factor

Feynman integral

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Feynman integrals

$$I_G = \int_{\mathbb{M}^{DL}} \frac{d^D k_1 \cdots d^D k_L}{\prod_e (q_e^2 - m_e^2 + i\varepsilon)}$$

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- Complicated and fascinating objects.

Feynman integrals

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- Hard to evaluate: There is no entirely general, effective evaluation algorithm.

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- Complicated and fascinating objects.
- Hard to evaluate: There is no entirely general, effective evaluation algorithm.
- Fastest algorithm works for finite, scalar, Euclidean Feynman integrals+:

Runtime (exponential): $\mathcal{O}(\,|E_G|\,2^{|E_G|} + \delta^{-2}\,|V_G|^3\,)$

MB 2020

Feynman integrals

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MB 2020

⇒ Evaluation (of 5-6 digits) is relatively fast up to ~17 loops

(algorithm heavily uses (tropical) **generalized permutahedra** geometry.)

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Even if the individual Feynman integrals are fast, there remains a

Major problem:

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⇒ Predictions take **factorial** computer-time

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Major problem: The number of graphs grows as ($L!$)

$\Rightarrow$ Predictions take **factorial** computer-time

… and ultimately become more expensive than measurement.

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⇒ Predictions take **factorial** computer-time

… and ultimately become more expensive than measurement.

From a **complexity theory** point of view QFT is not very good.

Anti—Slogan

Quantum field theory is also the worst theory.

It is very hard to actually make predictions.

$$\mathscr{A}_{L,n}(p_1, \ldots, p_n) = \sum_G \frac{1}{|\mathrm{Aut}(G)|} I_G$$

QFT observable

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Feynman integral

$$\mathcal{A}_{L,n}(p_1, \ldots, p_n) = \sum_G \frac{1}{|\mathrm{Aut}(G)|} I_G$$

Overarching idea to overcome this **factorial-time complexity** of QFT

Avoid individual Feynman graphs

See e.g.: Britto—Cachazo—Feng—Witten 2005

or Arkani-Hamed—Bai—Lam 2017

Tropical(ized) QFT

Tropical(ized) QFT

MB arXiv:2508.14263

Relation to previous work on **tropicalized Feynman integrals**

Schultka 2018, Panzer 2019, MB 2020, MB—Munch—Tellander 2023, MB-Fraaije 2025

Inspiration from and relations to **Surfaceology**

Arkani-Hamed, Figueiredo, Frost, Salvatori, Plamondon, Thomas, … 2023—

Particularly

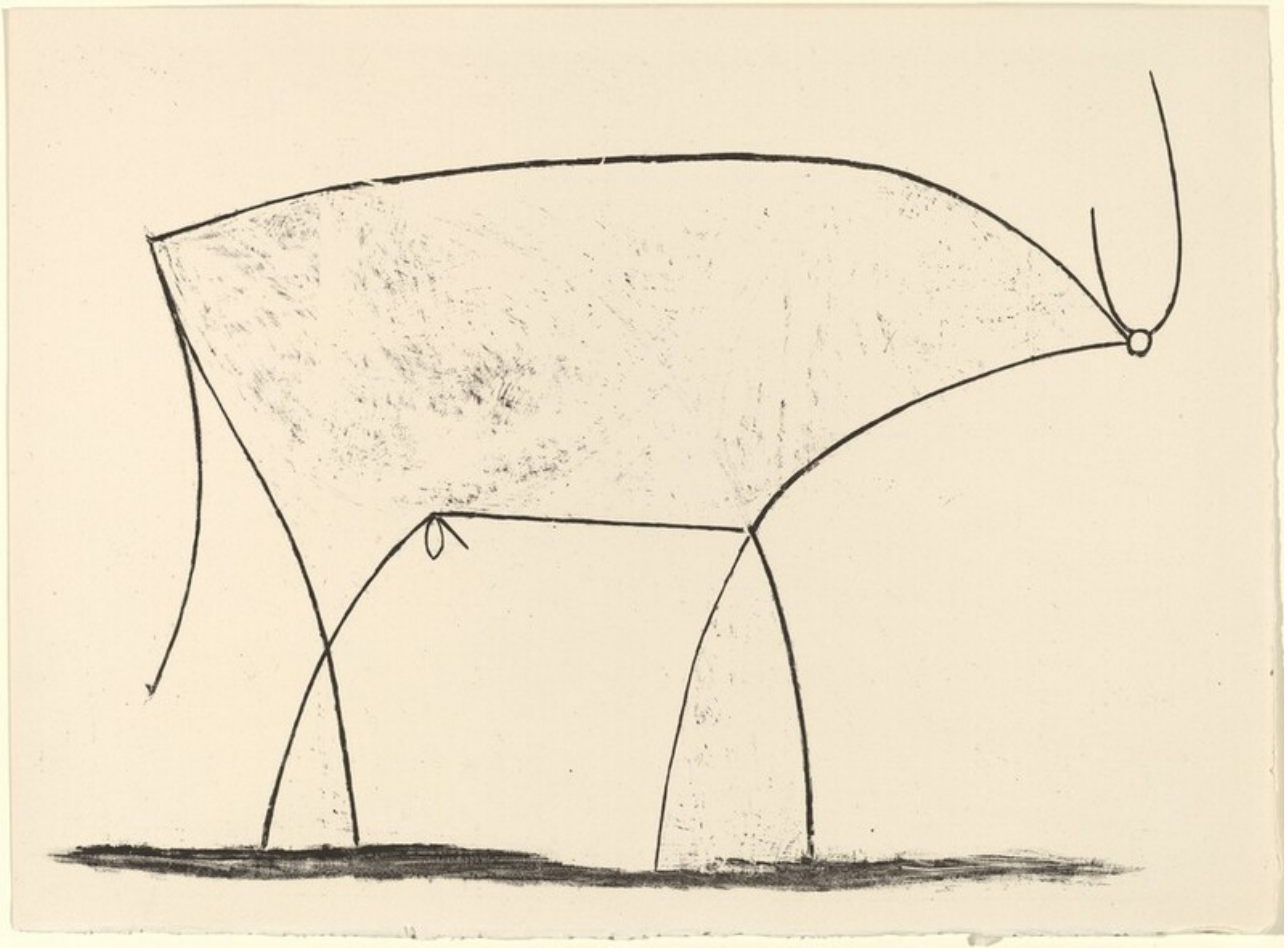
Salvatori 2025

Important empirical data:

Balduf 2023, Balduf—Shaban 2024, MB-Favorito 2025, Balduf—Thürigen 2025

Quantum field theory

Quantum field theory



Tropical quantum field theory

The geometry of Tropical QFT

QFT perturbation theory as an integration problem on the **moduli space of graphs**

$$\mathscr{A}_{L,n}(p_1, \ldots, p_n) = \sum_G \frac{1}{|\mathrm{Aut}(G)|} I_G$$

QFT observable

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Symmetry
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Feynman integral

$$\mathscr{A}_{L,n}(p_1, \ldots, p_n) = \sum_{G} \frac{1}{|\mathrm{Aut}(G)|} I_G$$

QFT observable

Sum over graphs with
L loops and n legs

Symmetry
factor

Feynman integral

Many shapes of Feynman integrals

$$I_G = (\star)\int_{\mathbb{M}^{DL}} \frac{d^D k_1 \cdots d^D k_L}{\prod_e (q_e^2 - m_e^2 + i\varepsilon)} = (\star)\int_{\mathbb{R}_{>0}^{E_G}} \exp(-F/U)\frac{d\alpha_1 \ldots d\alpha_{E_G}}{U^{D/2}} = \int_{\mathbb{R}_{>0}^{E_G}} \frac{d\alpha_1 \ldots d\alpha_{E_G}}{\mathscr{P}_G(\alpha, p)^{D/2}}$$

Here: Lee—Pomeransky polynomial/ representation

(i.e. Feynman/Schwinger/parametric representation)

$$\mathscr{A}_{L,n}(p_1, \ldots, p_n) = \sum_G \frac{1}{|\mathrm{Aut}(G)|} I_G$$

QFT observable

Sum over graphs with L loops and n legs

Symmetry factor

Feynman integral

$$\mathscr{A}_{L,n}(p_1,\ldots,p_n) = \sum_G \frac{1}{|\mathrm{Aut}G|} \int_{\mathbb{R}_{>0}^{E_G}} \frac{d\alpha_1 \ldots d\alpha_{E_G}}{\mathscr{P}_G(\boldsymbol{\alpha},\boldsymbol{p})^{D/2}}$$

1PI correlators /
Amplitudes

Sum over graphs with
L loops and n legs

Symmetry
factor

Rational integrand,
over positive orthand.

$$\mathscr{A}_{L,n}(p_1,\ldots,p_n) = \sum_G \frac{1}{|\mathrm{Aut}G|} \int_{\mathbb{R}_{>0}^{E_G}} \frac{d\alpha_1 \ldots d\alpha_{E_G}}{\mathscr{P}_G(\boldsymbol{\alpha},\boldsymbol{p})^{D/2}}$$

$$\mathscr{A}_{L,n}(p_1, \ldots, p_n) = \underbrace{\sum_G \frac{1}{|\mathrm{Aut}G|} \int_{\mathbb{R}^{E_G}_{>0}}}_{} \frac{d\alpha_1 \ldots d\alpha_{E_G}}{\mathscr{P}_G(\boldsymbol{\alpha}, \boldsymbol{p})^{D/2}}$$

Integral over a **moduli space of graphs.**

$$\mathscr{A}_{L,n}(p_1, \ldots, p_n) = \sum_G \frac{1}{|\mathrm{Aut}G|} \int_{\mathbb{R}_{>0}^{E_G}} \frac{d\alpha_1 \ldots d\alpha_{E_G}}{\mathscr{P}_G(\boldsymbol{\alpha}, \boldsymbol{p})^{D/2}}$$

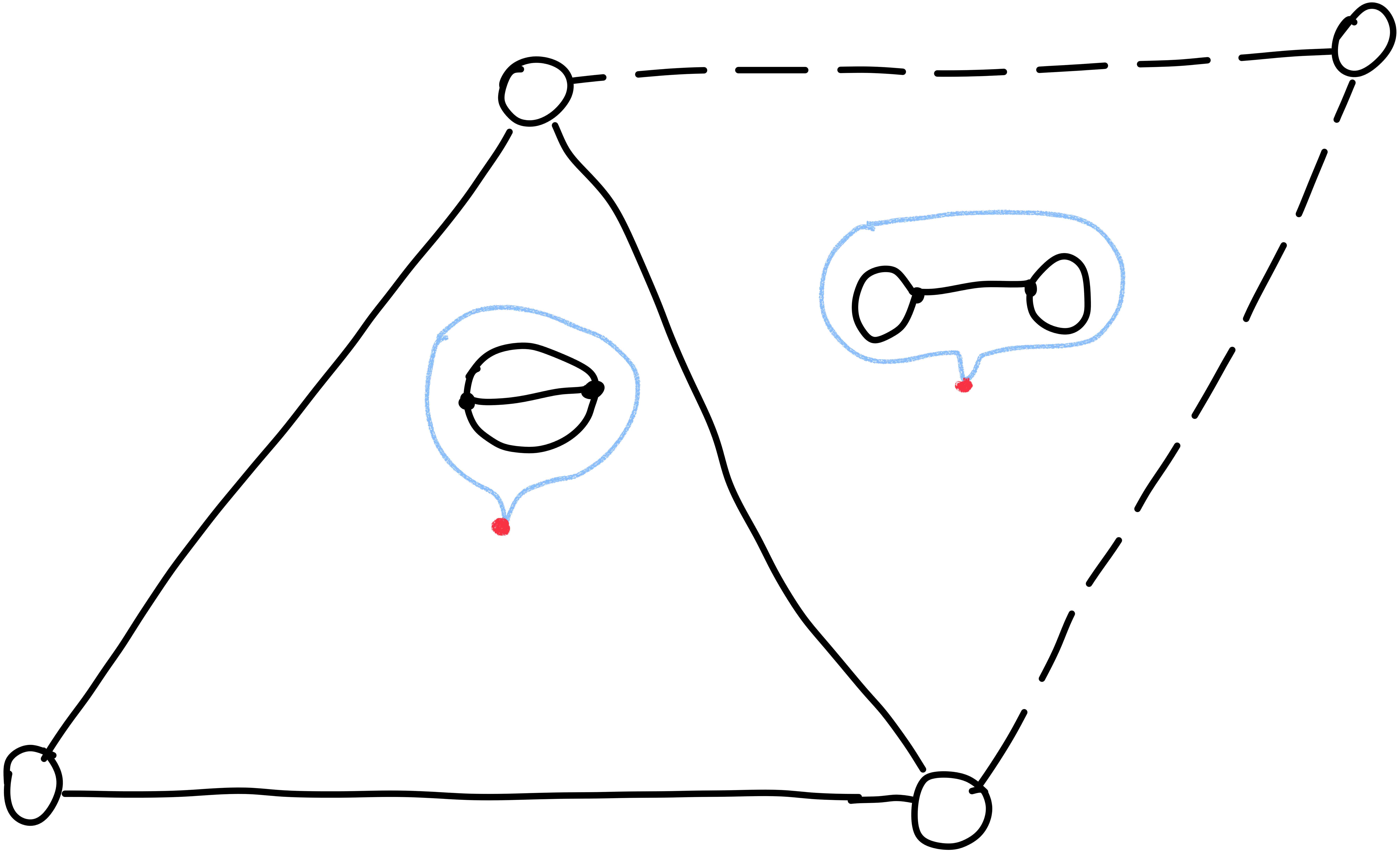
$$= \int_{\mathscr{M}_{L,n}^{\mathrm{tr}}} \mu_D(\boldsymbol{p})$$

where $\mathscr{M}_{L,n}^{\mathrm{tr}} = \bigsqcup_G \left(\mathbb{R}_{>0}^{E_G} \,/\, \mathrm{Aut}(G) \right)$

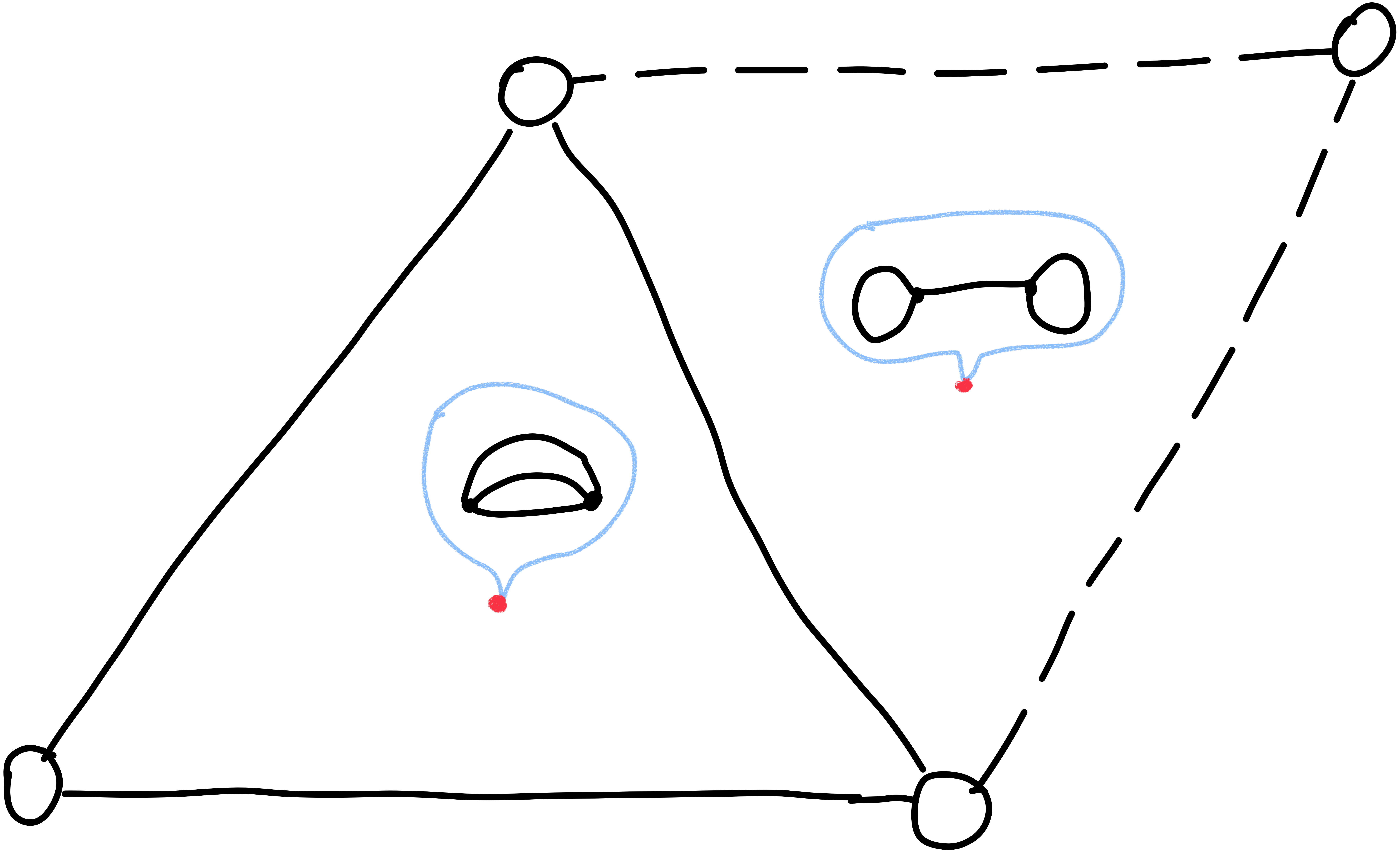
is (almost) the moduli space of graphs (and also the moduli space of tropical curves).

The moduli space of graphs

$\mathscr{M}^{\mathrm{tr}}_{2,0}$



$\mathscr{M}_{2,0}^{\mathrm{tr}}$



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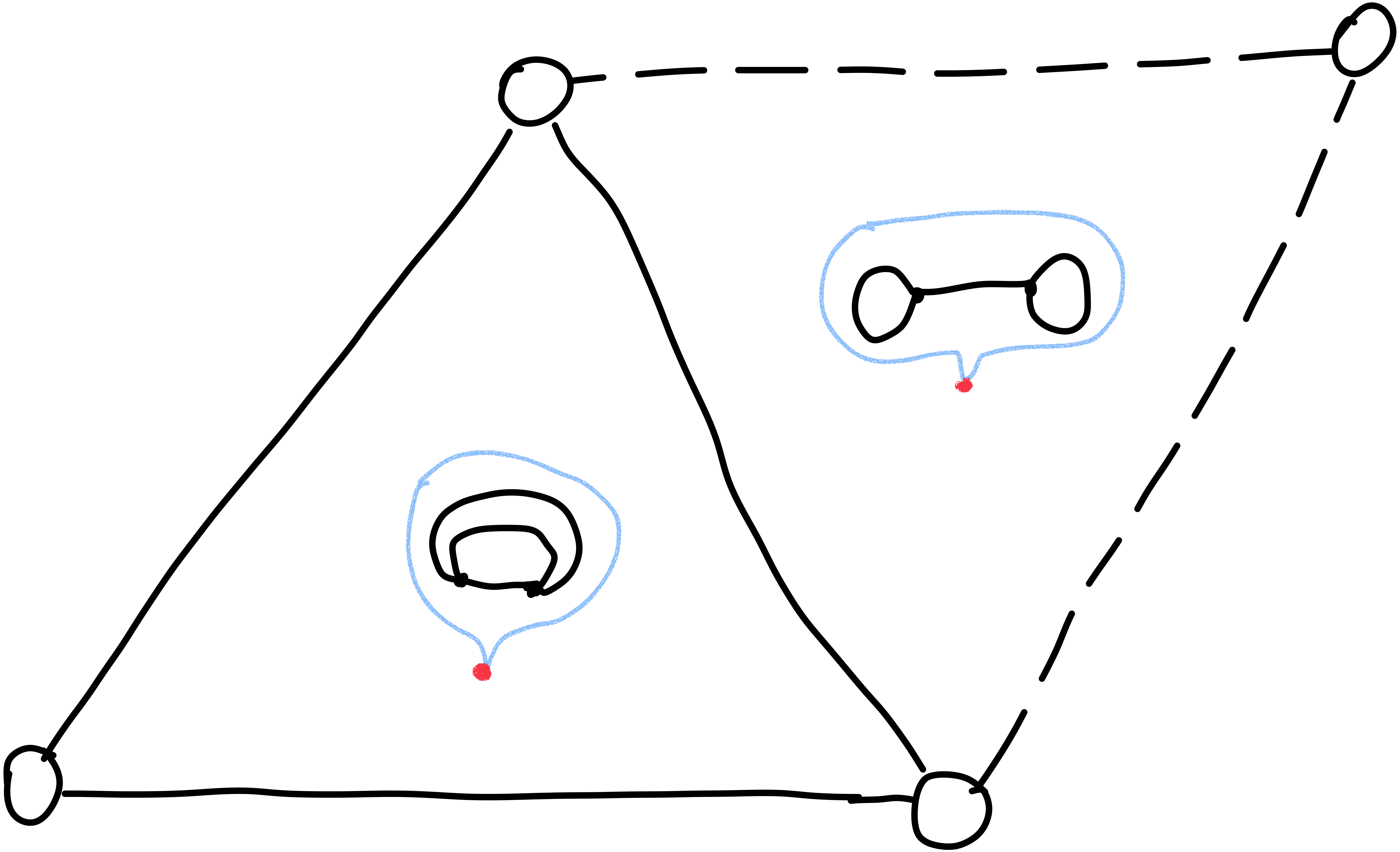

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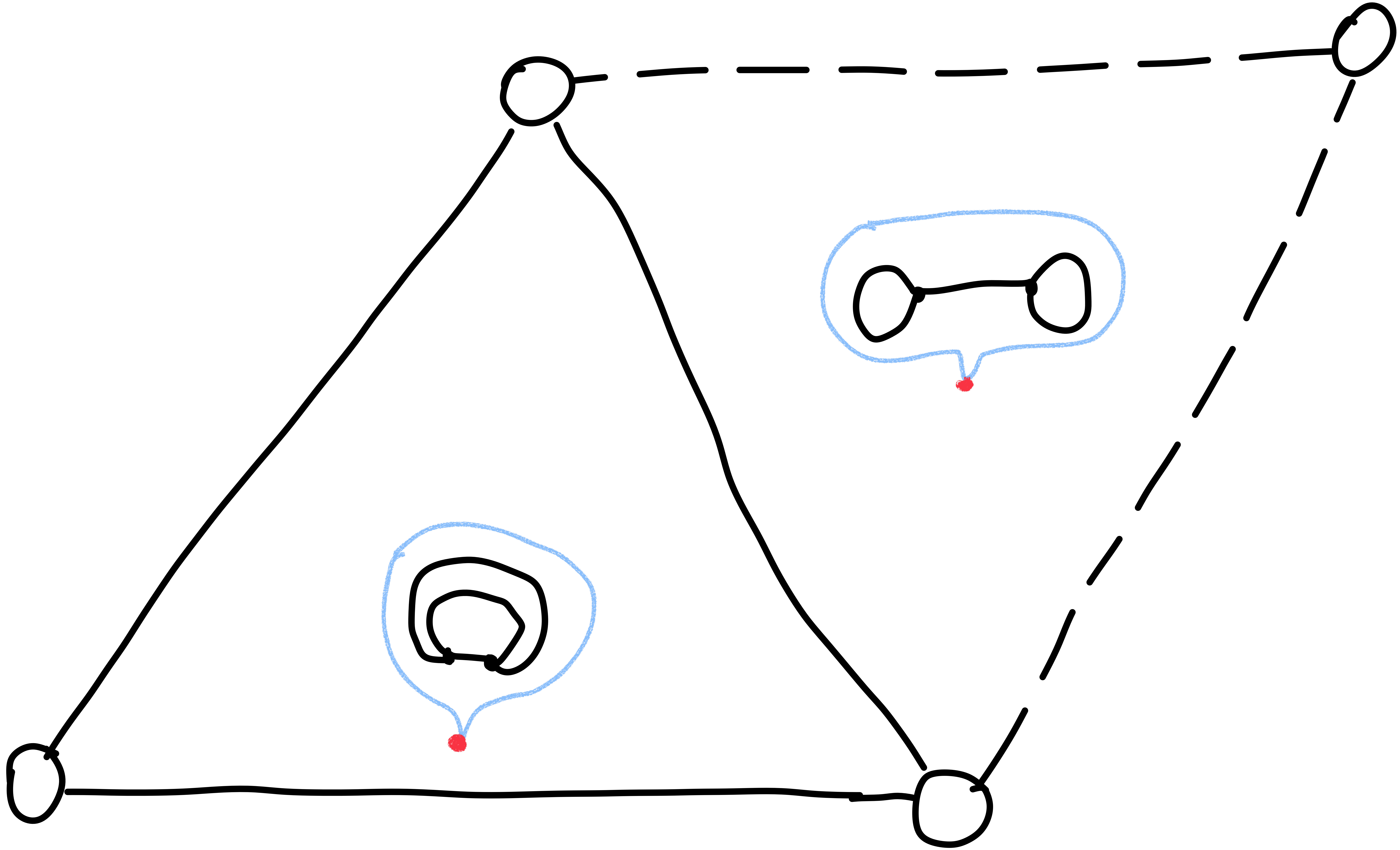
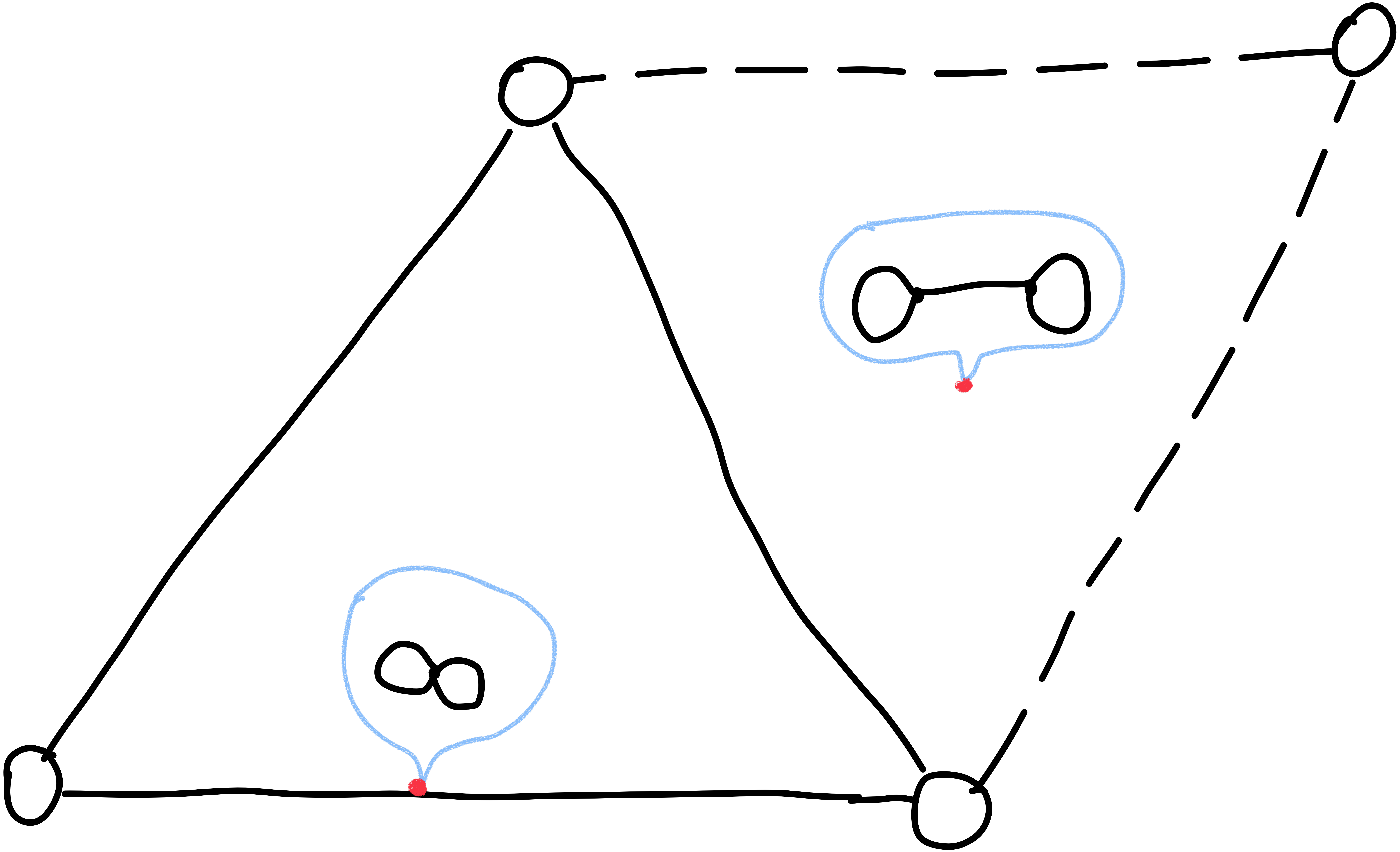
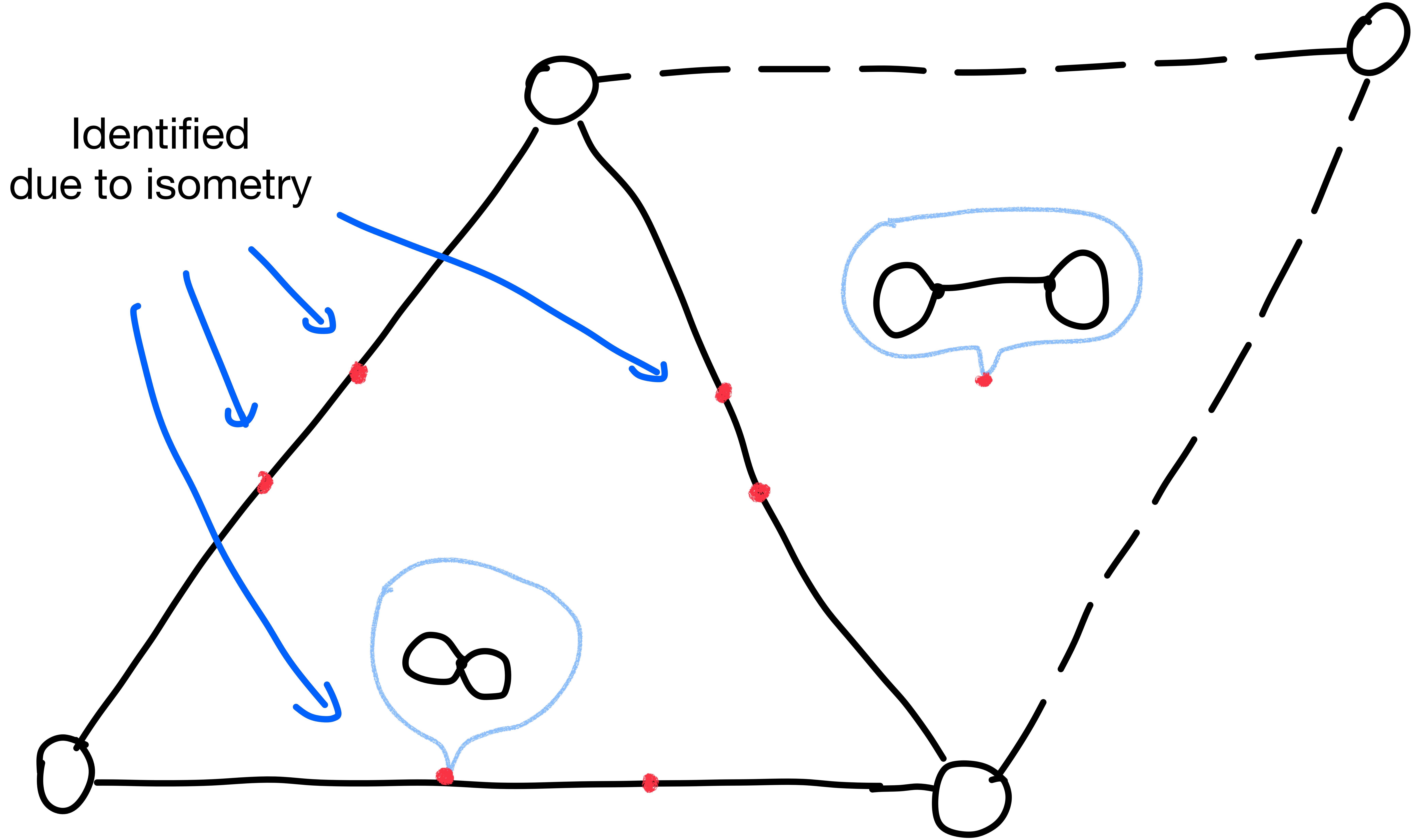


$\mathscr{M}_{2,0}^{\mathrm{tr}}$

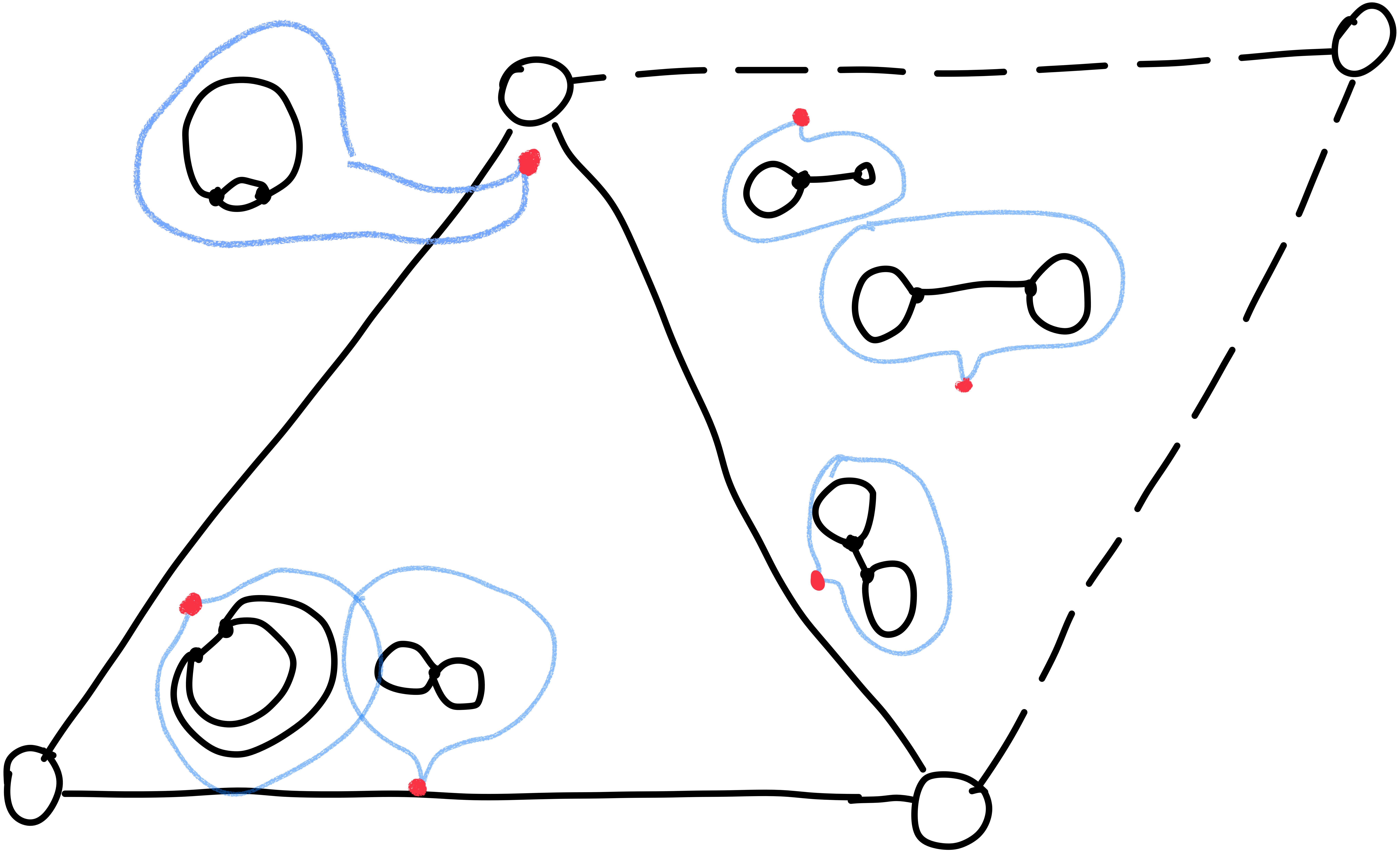


$\mathscr{M}^{\mathrm{tr}}_{2,0}$

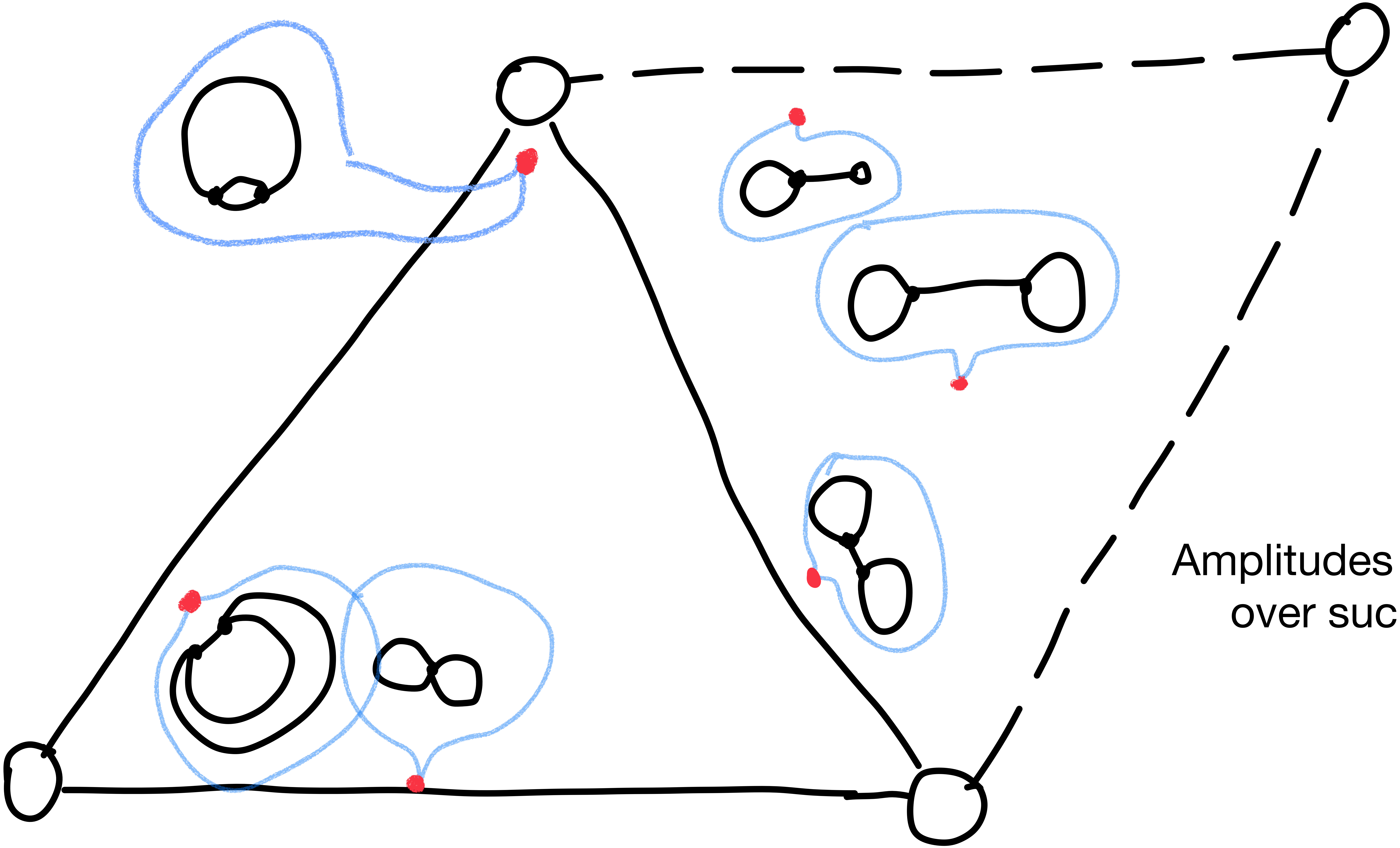


Identified
due to isometry

$\mathscr{M}^{\mathrm{tr}}_{2,0}$



$\mathscr{M}^{\mathrm{tr}}_{2,0}$



Amplitudes are integrals
over such spaces.

$$\mathscr{A}_{L,n}(p_1,\ldots,p_n) = \sum_G \frac{1}{|\mathrm{Aut}G|}\int_{\mathbb{R}^{E_G}_{>0}} \frac{d\alpha_1\ldots d\alpha_{E_G}}{\mathscr{P}_G(\boldsymbol{\alpha},\boldsymbol{p})^{D/2}}$$

$$= \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \mu_D(\boldsymbol{p})$$

Volume form on the moduli space.
(Takes form of Feynman integrand on each graph stratum)

$$\mathscr{A}_{L,n}(p_1, \ldots, p_n) = \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \mu_D(\boldsymbol{p})$$

Analogy to Mirzakhani's volume recursion on moduli spaces of hyperbolic surfaces

(This volume recursion also plays a key role in **surfaceology**.)

$$V_{g,n}(L_1, \ldots, L_n) = \int_{\mathscr{M}_{g,n}(L_1,\ldots,L_n)} \omega_{WP}$$

Weil—Petersson form

Moduli space of hyperbolic surfaces of genus g with boundaries of length $L_1, \ldots, L_n$.

$$V_{g,n}(L_1, \ldots, L_n) = \int_{\mathscr{M}_{g,n}(L_1,\ldots,L_n)} \omega_{WP}$$

Weil—Petersson form

Moduli space of hyperbolic surfaces of genus g with boundaries of length $L_1, \ldots, L_n$.

Theorem

Mirzakhani 2007

$V_{g,n}(L_1, \ldots, L_n)$ respects **recursion** equations (over all g, n).

⇒ The volumes $V_{g,n}(L_1, \ldots, L_n)$ can be computed easily.

⇒ Allows the study properties of **random surfaces** distributed as ω_{WP}.

Can we find a similar recursion for amplitudes?

$$\mathscr{A}_{L,n}(p_1, \ldots, p_n) = \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \mu_D(\boldsymbol{p})$$

For $L = 0$, yes → **Berens—Giele 1988**, **BCFW 2005, …**

For $L > 0$, no…

(Likely impossible by **Belkale—Brosnan 2000**
and **Mnëv** universality **1985**)

Tropicalize…

$$\mathscr{A}_{L,n}(p_1, \ldots, p_n) = \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \mu_D(\boldsymbol{p})$$

Tropicalized amplitudes

$$\mathscr{A}^{\mathrm{tr}}_{L,n}(p_1, \ldots, p_n) = \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \mu^{\mathrm{tr}}_{D}(\boldsymbol{p})$$

Tropicalized amplitudes

$$\mathscr{A}^{\mathrm{tr}}_{L,n}(p_1,\ldots,p_n) = \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \mu^{\mathrm{tr}}_D(\boldsymbol{p})$$

Theorem

MB 2025

$\mathscr{A}^{\mathrm{tr}}_{L,n}$ respects recursion equations (over all L, n).

$\Rightarrow$ The amplitudes $\mathscr{A}^{\mathrm{tr}}_{L,n}(p_1,\ldots,p_n)$ can be computed easily.

$\Rightarrow$ We can study properties of **random graphs** distributed as $\mu^{\mathrm{tr}}_D(\boldsymbol{p})$.

Tropicalization

Polynomial $$P(x_1, \ldots, x_n) = \sum_{\boldsymbol{k} \in M} a_{\boldsymbol{k}} \prod_{i=1}^{n} x_i^{k_i} \quad \in \mathbb{R}_+[x_1, \ldots, x_n]$$

Tropicalization

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Definition
‘Tropical approximation’

$$P^{\mathrm{tr}}(x_1, \ldots, x_n) := \max_{\boldsymbol{k} \in M} \prod_{i=1}^{n} x_i^{k_i}$$

Rule: $\sum \to \max$ and $a_{\boldsymbol{k}} \to 1.$

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'Typical' tropicalization is recovered in $\log$ coordinates.

Tropicalization

Polynomial $$P(x_1, \ldots, x_n) = \sum_{\boldsymbol{k} \in M} a_{\boldsymbol{k}} \prod_{i=1}^{n} x_i^{k_i} \in \mathbb{R}_+[x_1, \ldots, x_n]$$

Definition
'Tropical approximation'

$$P^{\mathrm{tr}}(x_1, \ldots, x_n) := \max_{\boldsymbol{k} \in M} \prod_{i=1}^{n} x_i^{k_i}$$

Lemma

There are $C_1, C_2 > 0$, such that $C_1 \leq \dfrac{P(x)}{P^{\mathrm{tr}}(x)} \leq C_2$ for all $x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$.

$\Rightarrow$ The tropicalization provides an **envelope** for the original object.

Tropicalized amplitudes

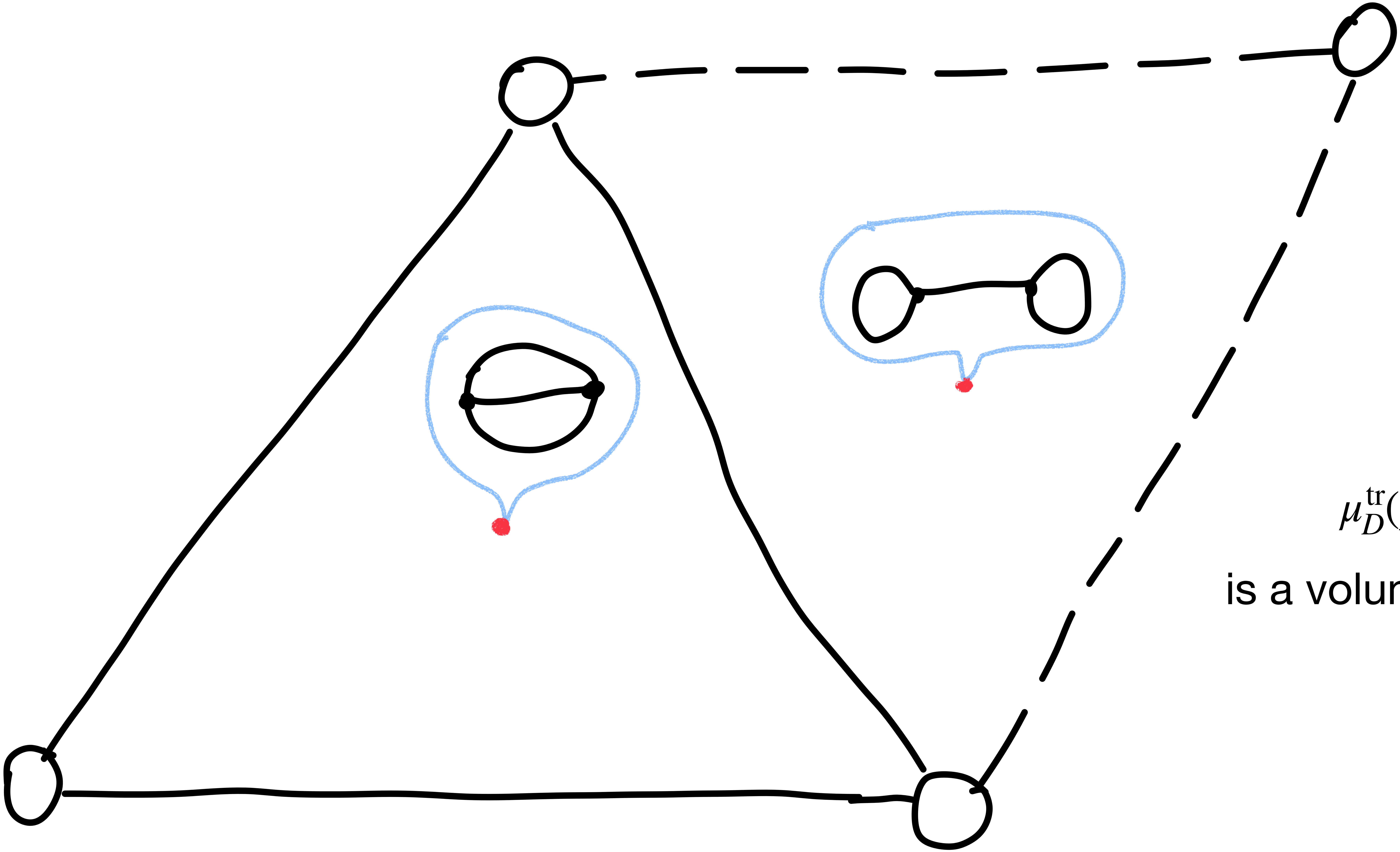
$$\mathscr{A}^{\mathrm{tr}}_{L,n}(p_1, \ldots, p_n) = \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \mu^{\mathrm{tr}}_{D}(\boldsymbol{p})$$

Tropicalized amplitudes

$$\mathscr{A}^{\mathrm{tr}}_{L,n}(p_1, \ldots, p_n) = \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \mu^{\mathrm{tr}}_D(\boldsymbol{p})$$

$$\mu_D(\boldsymbol{p})\,|_{\Delta_G} = \frac{d\alpha_1 \ldots d\alpha_{E_G}}{\mathscr{P}_G(\boldsymbol{\alpha}, \boldsymbol{p})^{D/2}} \quad \Rightarrow \quad \mu^{\mathrm{tr}}_D(\boldsymbol{p})\,|_{\Delta_G} = \frac{d\alpha_1 \ldots d\alpha_{E_G}}{\mathscr{P}^{\mathrm{tr}}_G(\boldsymbol{\alpha}, \boldsymbol{p})^{D/2}}$$

$\mathscr{M}_{2,0}^{\mathrm{tr}}$



$\mu_D^{\mathrm{tr}}(\boldsymbol{p})$

is a volume form

Lagrangian interpretation of tropicalization

Massive scalar quantum field theory

$$V[\Phi] = \sum_{k \geq 3} \lambda_k \Phi^k / k!$$

Action $$\mathcal{S}[\Phi, J] = \int_{\mathbb{R}^D} \mathrm{d}^D x \left(\frac{1}{2} \Phi(x) \left(\Box + m^2 \right) \Phi(x) - V[\Phi](x) - J(x)\Phi(x) \right)$$

Massive scalar quantum field theory

$$V[\Phi] = \sum_{k\geq 3} \lambda_k \Phi^k / k!$$

Action $$\mathcal{S}^{\xi}[\Phi, J] = \int_{\mathbb{R}^{D\cdot\xi}} \mathrm{d}^{D\cdot\xi}x \left(\frac{1}{2}\Phi(x)\left(\Box + m^2\right)^{\xi}\Phi(x) - V[\Phi](x) - J(x)\Phi(x) \right)$$

Deformation: $D \to D \cdot \xi$ and $(\Box + m^2) \to (\Box + m^2)^{\xi}$

(Relative mass scalings of all operators remain constant.)

Massive scalar quantum field theory

$$V[\Phi] = \sum_{k \geq 3} \lambda_k \Phi^k / k!$$

Action

$$\mathcal{S}^{\xi}[\Phi, J] = \int_{\mathbb{R}^{D \cdot \xi}} \mathrm{d}^{D \cdot \xi} x \left(\frac{1}{2} \Phi(x) \left(\Box + m^2 \right)^{\xi} \Phi(x) - V[\Phi](x) - J(x)\Phi(x) \right),$$

Partition function

$$Z^{\xi}[J] = \int \exp\left(-\mathcal{S}^{\xi}[\Phi, J] \right) \mathcal{D}[\Phi]$$

Massive scalar quantum field theory

$$V[\Phi] = \sum_{k\geq 3} \lambda_k \Phi^k / k!$$

Action

$$\mathcal{S}^{\xi}[\Phi, J] = \int_{\mathbb{R}^{D\cdot\xi}} \mathrm{d}^{D\cdot\xi}x \left(\frac{1}{2}\Phi(x)\left(\Box + m^2\right)^{\xi}\Phi(x) - V[\Phi](x) - J(x)\Phi(x)\right);$$

Partition function

$$Z^{\xi}[J] = \int \exp\left(-\mathcal{S}^{\xi}[\Phi, J]\right) \mathcal{D}[\Phi]$$

Initial QFT

$$\xi = 1$$

→

Theorem MB 2025

$$\lim_{\xi\to 0^+} Z^{\xi} = Z^{\mathrm{tr}}$$

Tropical QFT is a **deformation limit** of the initial QFT.

The **combinatorics** of Tropical QFT

Effective action (a generating function of tropicalized amplitudes):

$$\Gamma^{\mathrm{tr}}(\hbar, \varphi) = \sum_{L,n} \mathscr{A}^{\mathrm{tr}}_{L,n} \hbar^L \varphi^n$$

Recursive solution ↔ exact solvability

Theorem (MB 2025)

The tropical QFT's effective action is completely fixed by the nonlinear PDE

$$\mathscr{P}_D \Gamma^{\mathrm{tr}} = \left(1 - \frac{\partial^2 \Gamma^{\mathrm{tr}}}{\partial \varphi^2} \right)^{-1} - 1, \qquad \text{where}$$

Recursive solution ↔ exact solvability

Theorem **(MB 2025)**

The tropical QFT's effective action is completely fixed by the nonlinear PDE

$$\mathscr{P}_D \Gamma^{\mathrm{tr}} = \left(1 - \frac{\partial^2 \Gamma^{\mathrm{tr}}}{\partial \varphi^2}\right)^{-1} - 1, \qquad \text{where}$$

$$\mathcal{P}_D = \left(-D - \left(1 - \frac{D}{2}\right)\varphi \frac{\partial}{\partial \varphi} + \sum_{k \geq 3} \left(k - D\left(\frac{k}{2} - 1\right)\right)\lambda_k \frac{\partial}{\partial \lambda_k}\right)$$

(+ trivial boundary conditions.)

Proof of the solvability/recursion

Exploits a recursive relation between tropical Feynman integrals/Hepp bounds

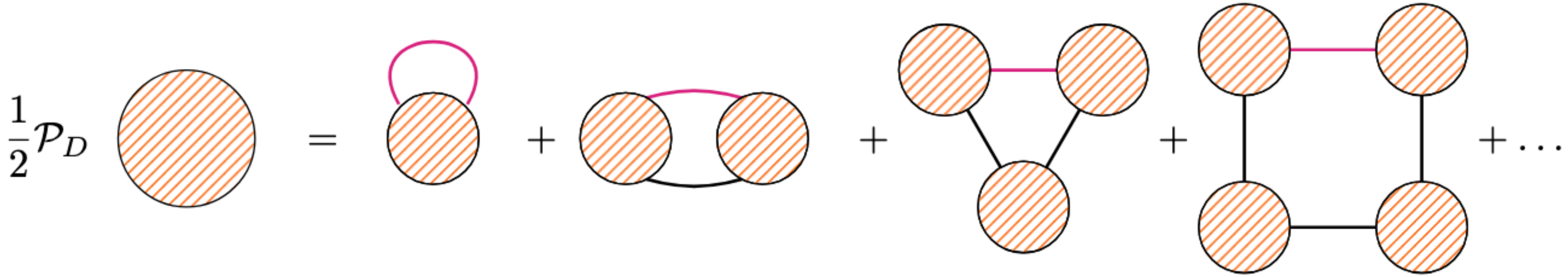


Figure 3: Illustration of the tropical loop equation. Each orange blob stands for a 1PI graph. The right-hand graphs have one purple pointed edge. Cutting these edges yields beaded graphs.

Tropicalized amplitudes

$$\mathscr{A}^{\mathrm{tr}}_{L,n}(p_1, \ldots, p_n) = \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \mu^{\mathrm{tr}}_D(\boldsymbol{p})$$

Theorem

MB 2025

$\mathscr{A}^{\mathrm{tr}}_{L,n}$ respects recursion equations (over all L, n).

⇒ The amplitudes $\mathscr{A}^{\mathrm{tr}}_{L,n}(p_1, \ldots, p_n)$ can be computed easily.

⇒ We can study properties of **random graphs** distributed as $\mu^{\mathrm{tr}}_D(\boldsymbol{p})$.

Using tropical QFT for **polynomial-time** perturbation theory in non-tropical QFT

…and solving the **factorial-time problem** of QFT perturbation theory.

Recall…

$$\mathscr{A}_{L,n}(p_1, \ldots, p_n) = \int_{\mathscr{M}_{L,n}^{\mathrm{tr}}} \mu_D(\boldsymbol{p})$$

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Recall…

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$$\mathscr{A}_{L,n}^{\mathrm{tr}}(p_1, \ldots, p_n) = \int_{\mathscr{M}_{L,n}^{\mathrm{tr}}} \mu_D^{\mathrm{tr}}(\boldsymbol{p})$$

$$\mu_D(\boldsymbol{p})\,|_{\Delta_G} = \frac{d\alpha_1 \ldots d\alpha_{E_G}}{\mathscr{P}_G(\boldsymbol{\alpha}, \boldsymbol{p})^{D/2}} \qquad \mu_D^{\mathrm{tr}}(\boldsymbol{p})\,|_{\Delta_G} = \frac{d\alpha_1 \ldots d\alpha_{E_G}}{\mathscr{P}_G^{\mathrm{tr}}(\boldsymbol{\alpha}, \boldsymbol{p})^{D/2}}$$

$$\mathscr{A}_{L,n}(p_1, \ldots, p_n) = \int_{\mathscr{M}_{L,n}^{\mathrm{tr}}} \mu_D(\boldsymbol{p})$$

$$\begin{aligned}\mathscr{A}_{L,n}(p_1,\ldots,p_n) &= \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \mu_D(\boldsymbol{p}) \\ &= \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \left(\frac{\mathscr{P}^{\mathrm{tr}}_G(\boldsymbol{\alpha},\boldsymbol{p})}{\mathscr{P}_G(\boldsymbol{\alpha},\boldsymbol{p})}\right)^{D/2} \mu^{\mathrm{tr}}_D(\boldsymbol{p})\end{aligned}$$

$$\begin{aligned}
\mathscr{A}_{L,n}(p_1,\ldots,p_n) &= \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \mu_D(\boldsymbol{p}) \\
&= \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \left(\frac{\mathscr{P}^{\mathrm{tr}}_G(\boldsymbol{\alpha},\boldsymbol{p})}{\mathscr{P}_G(\boldsymbol{\alpha},\boldsymbol{p})}\right)^{D/2} \mu^{\mathrm{tr}}_D(\boldsymbol{p}) \\
&= \mathscr{A}^{\mathrm{tr}}_{L,n} \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \left(\frac{\mathscr{P}^{\mathrm{tr}}_G(\boldsymbol{\alpha},\boldsymbol{p})}{\mathscr{P}_G(\boldsymbol{\alpha},\boldsymbol{p})}\right)^{D/2} \widehat{\mu}^{\mathrm{tr}}_D(\boldsymbol{p})
\end{aligned}$$

Key: this term is bounded.

Probability density

$$\mathscr{A}_{L,n}(p_1,\ldots,p_n) = \mathscr{A}^{\mathrm{tr}}_{L,n}\int_{\mathscr{M}^{\mathrm{tr}}_{L,n}}\left(\frac{\mathscr{P}^{\mathrm{tr}}_G(\boldsymbol{\alpha},\boldsymbol{p})}{\mathscr{P}_G(\boldsymbol{\alpha},\boldsymbol{p})}\right)^{D/2}\widehat{\mu}^{\mathrm{tr}}_D(\boldsymbol{p})$$

Key: this term is bounded.

Numerical evaluation of $\mathscr{A}_{L,n}(p_1,\ldots,p_n)$

Monte Carlo

Sampling algorithms for $\int_{\mathscr{M}^{\mathrm{tr}}_{L,n}}\widehat{\mu}^{\mathrm{tr}}_D(\boldsymbol{p}) = 1.$

Theorem

MB 2025

There is a **global sampling** algorithm for the measure $\widehat{\mu}_D^{\mathrm{tr}}(\boldsymbol{p})$ on $\mathscr{M}_{L,n}^{\mathrm{tr}}$.

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It's runtime and memory requirements scale **polynomially** with L, n .

Proof: uses exact solution of tropical QFT.

$\Rightarrow \mathscr{A}_{L,n}(p_1, \ldots, p_n)$ can be be estimated to finite accuracy in **polynomial-time** in L and n.

Fastest algorithms for Feynman integration run in exponential time.

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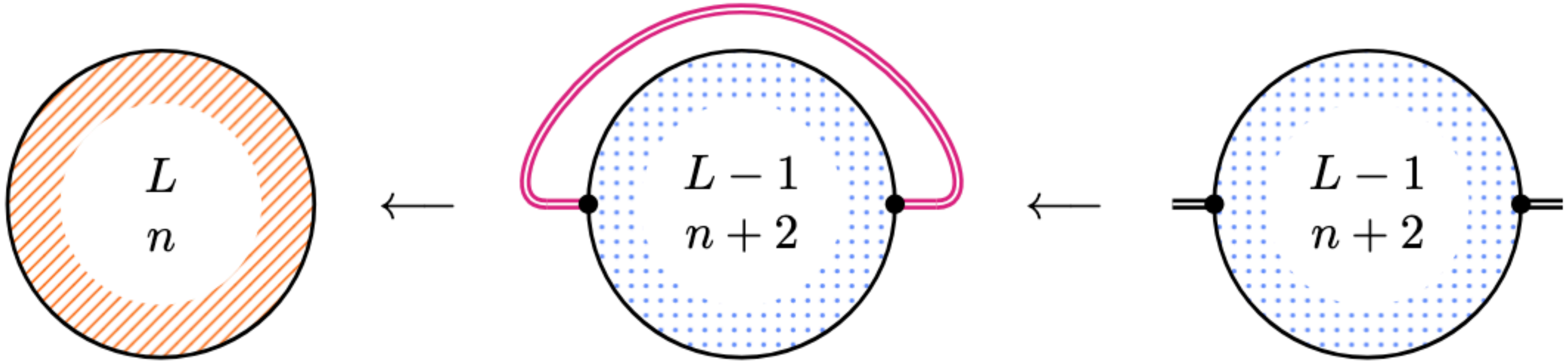
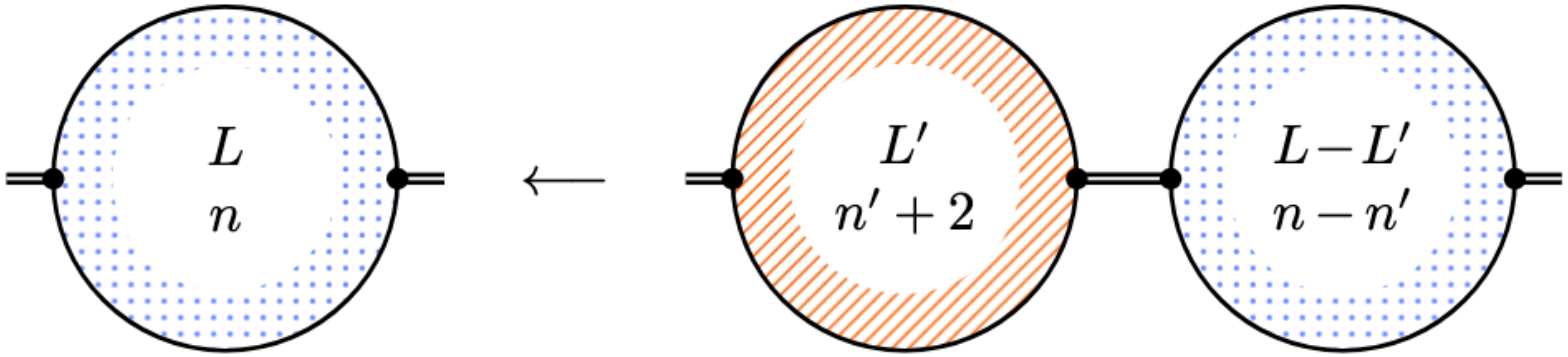
⇒ Evaluation of **individual Feynman integrals** quickly becomes slower than evaluating the **whole amplitude** $\mathscr{A}_{L,n}(p_1, \ldots, p_n)$.



(a) Illustration of the $L > 0$ case of Algorithm 18.



(b) Illustration of the (L', n') case in Algorithm 19.

The algorithm recursively produces metric graphs with the correct probability distribution. Most graphs will never be generated.

Polynomial-time bottleneck

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$$\mathscr{A}_{L,n}(p_1,\ldots,p_n) = \mathscr{A}^{\mathrm{tr}}_{L,n} \int_{\mathscr{M}^{\mathrm{tr}}_{L,n}} \left(\frac{\mathscr{P}^{\mathrm{tr}}_{G}(\boldsymbol{\alpha},\boldsymbol{p})}{\mathscr{P}_{G}(\boldsymbol{\alpha},\boldsymbol{p})} \right)^{D/2} \widehat{\mu}^{\mathrm{tr}}_{D}(\boldsymbol{p})$$

The bottleneck is the evaluation of the integrand polynomial $\mathscr{P}_G(\boldsymbol{\alpha},\boldsymbol{p})$.

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The bottleneck is the evaluation of the integrand polynomial $\mathscr{P}_G(\boldsymbol{\alpha}, \boldsymbol{p})$.

This computation can be reduced to manipulations of a graph Laplacian matrix.

Algorithms for such manipulations are among the fastest known for their problem class in theoretical computer science: **Spielman—Teng 2008**

Example computations

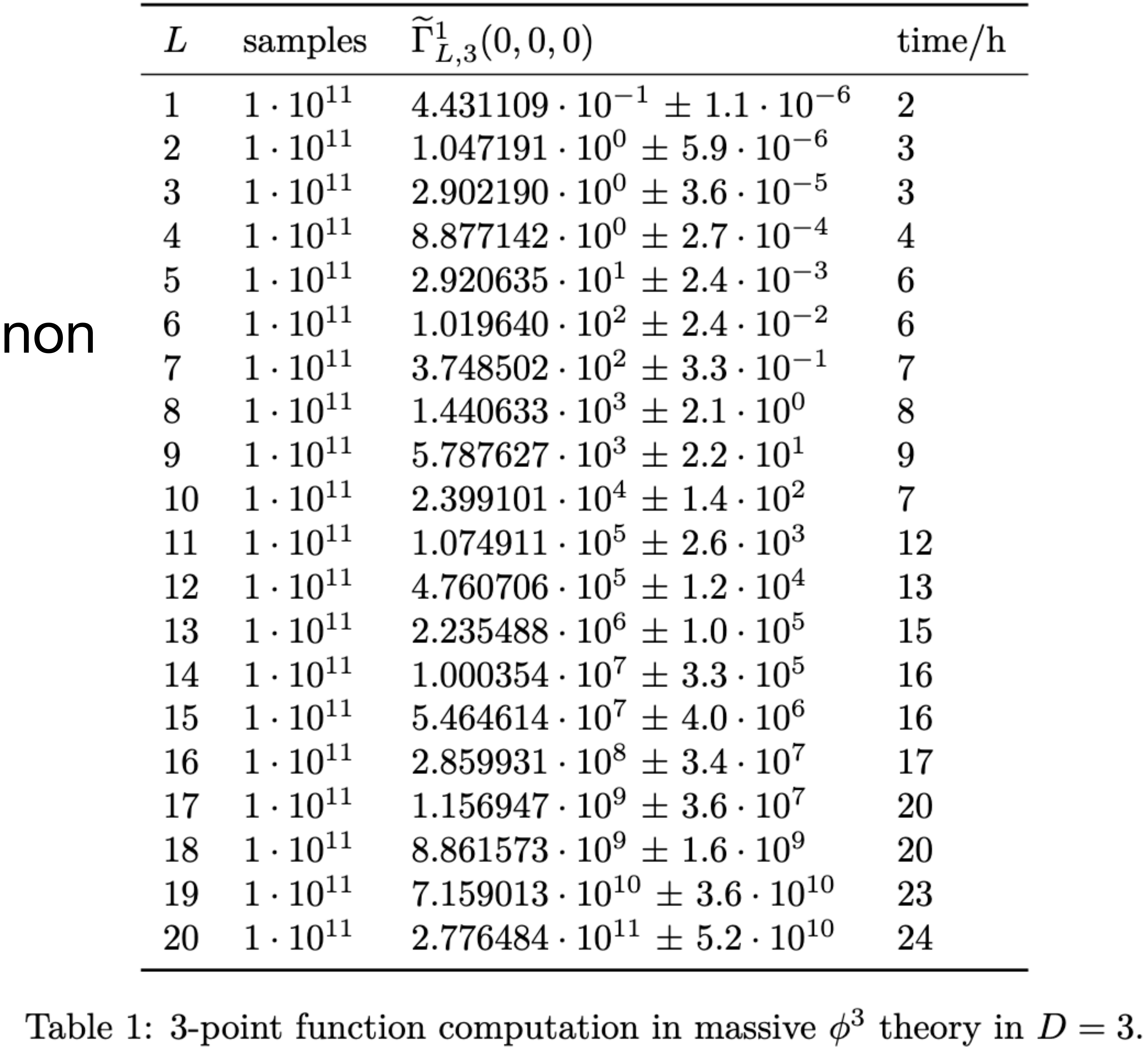
Massive ϕ^3 theory in $D = 3$, with $m^2 = 1$.

Relations to **Lee—Yang 1952** phenomenon

Known (numerically) up to $L \leq 7$

(**Sberveglieria, Spada 2024**)

Example computation for $L \leq 20$ using global tropical sampling



L	samples	$\widetilde{\Gamma}^1_{L,3}(0,0,0)$	time/h
1	$1 \cdot 10^{11}$	$4.431109 \cdot 10^{-1} \pm 1.1 \cdot 10^{-6}$	2
2	$1 \cdot 10^{11}$	$1.047191 \cdot 10^{0} \pm 5.9 \cdot 10^{-6}$	3
3	$1 \cdot 10^{11}$	$2.902190 \cdot 10^{0} \pm 3.6 \cdot 10^{-5}$	3
4	$1 \cdot 10^{11}$	$8.877142 \cdot 10^{0} \pm 2.7 \cdot 10^{-4}$	4
5	$1 \cdot 10^{11}$	$2.920635 \cdot 10^{1} \pm 2.4 \cdot 10^{-3}$	6
6	$1 \cdot 10^{11}$	$1.019640 \cdot 10^{2} \pm 2.4 \cdot 10^{-2}$	6
7	$1 \cdot 10^{11}$	$3.748502 \cdot 10^{2} \pm 3.3 \cdot 10^{-1}$	7
8	$1 \cdot 10^{11}$	$1.440633 \cdot 10^{3} \pm 2.1 \cdot 10^{0}$	8
9	$1 \cdot 10^{11}$	$5.787627 \cdot 10^{3} \pm 2.2 \cdot 10^{1}$	9
10	$1 \cdot 10^{11}$	$2.399101 \cdot 10^{4} \pm 1.4 \cdot 10^{2}$	7
11	$1 \cdot 10^{11}$	$1.074911 \cdot 10^{5} \pm 2.6 \cdot 10^{3}$	12
12	$1 \cdot 10^{11}$	$4.760706 \cdot 10^{5} \pm 1.2 \cdot 10^{4}$	13
13	$1 \cdot 10^{11}$	$2.235488 \cdot 10^{6} \pm 1.0 \cdot 10^{5}$	15
14	$1 \cdot 10^{11}$	$1.000354 \cdot 10^{7} \pm 3.3 \cdot 10^{5}$	16
15	$1 \cdot 10^{11}$	$5.464614 \cdot 10^{7} \pm 4.0 \cdot 10^{6}$	16
16	$1 \cdot 10^{11}$	$2.859931 \cdot 10^{8} \pm 3.4 \cdot 10^{7}$	17
17	$1 \cdot 10^{11}$	$1.156947 \cdot 10^{9} \pm 3.6 \cdot 10^{7}$	20
18	$1 \cdot 10^{11}$	$8.861573 \cdot 10^{9} \pm 1.6 \cdot 10^{9}$	20
19	$1 \cdot 10^{11}$	$7.159013 \cdot 10^{10} \pm 3.6 \cdot 10^{10}$	23
20	$1 \cdot 10^{11}$	$2.776484 \cdot 10^{11} \pm 5.2 \cdot 10^{10}$	24

Table 1: 3-point function computation in massive ϕ^3 theory in $D = 3$.

Primitive β function in 4-dim ϕ^4 theory

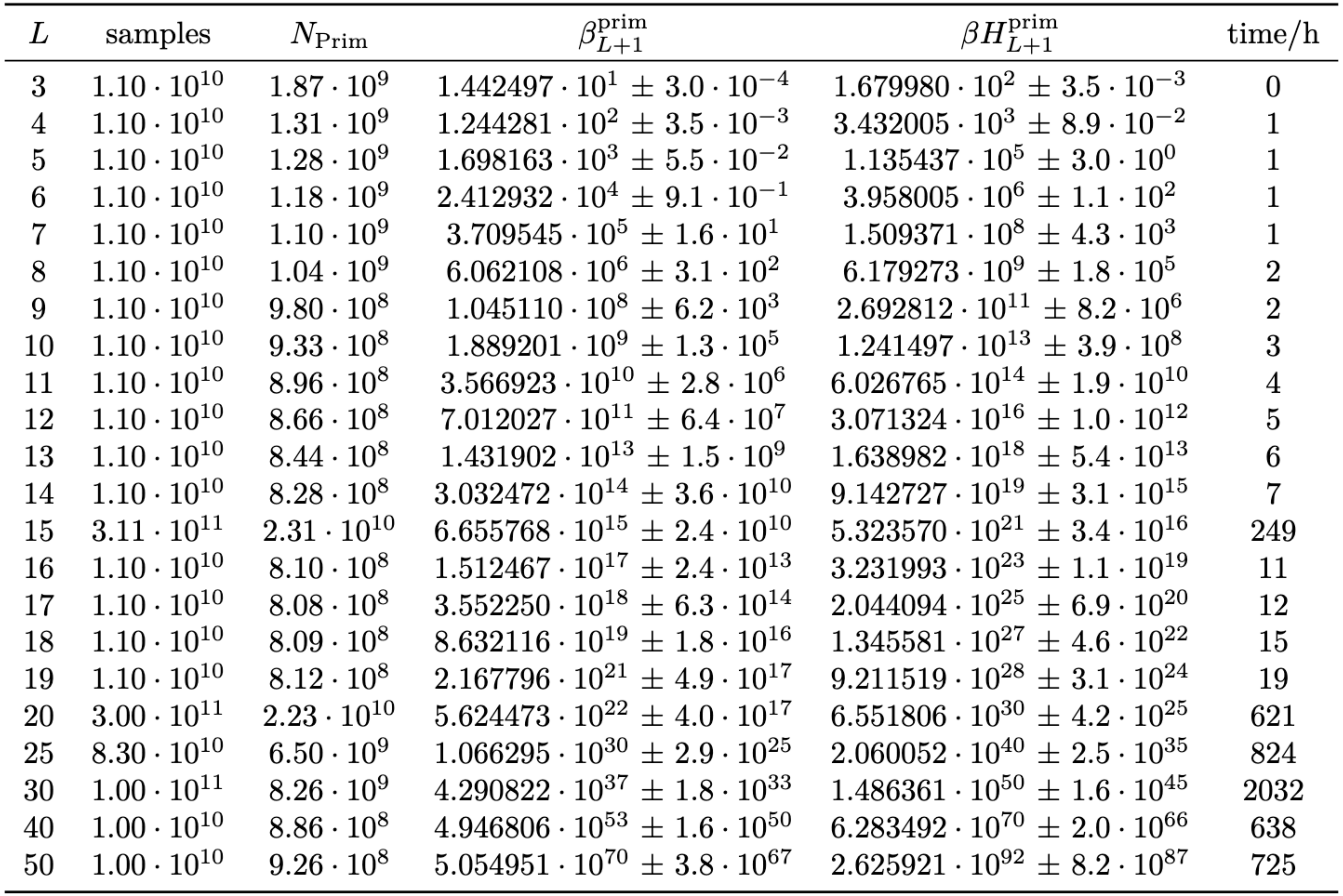
(Conjectured to be equal to full MS β function for large L.)

Relation to the **Ising model**

Known before (analytically) for $L \leq 7$ **Schnetz 2022.**

With tropical sampling (numerically) up to $L \leq 18$ **Balduf 2023**.

Example computation for $L \leq 50$ using global tropical sampling



L	samples	N_{Prim}	$\beta^{\text{prim}}_{L+1}$	$\beta H^{\text{prim}}_{L+1}$	time/h
3	$1.10 \cdot 10^{10}$	$1.87 \cdot 10^{9}$	$1.442497 \cdot 10^{1} \pm 3.0 \cdot 10^{-4}$	$1.679980 \cdot 10^{2} \pm 3.5 \cdot 10^{-3}$	0
4	$1.10 \cdot 10^{10}$	$1.31 \cdot 10^{9}$	$1.244281 \cdot 10^{2} \pm 3.5 \cdot 10^{-3}$	$3.432005 \cdot 10^{3} \pm 8.9 \cdot 10^{-2}$	1
5	$1.10 \cdot 10^{10}$	$1.28 \cdot 10^{9}$	$1.698163 \cdot 10^{3} \pm 5.5 \cdot 10^{-2}$	$1.135437 \cdot 10^{5} \pm 3.0 \cdot 10^{0}$	1
6	$1.10 \cdot 10^{10}$	$1.18 \cdot 10^{9}$	$2.412932 \cdot 10^{4} \pm 9.1 \cdot 10^{-1}$	$3.958005 \cdot 10^{6} \pm 1.1 \cdot 10^{2}$	1
7	$1.10 \cdot 10^{10}$	$1.10 \cdot 10^{9}$	$3.709545 \cdot 10^{5} \pm 1.6 \cdot 10^{1}$	$1.509371 \cdot 10^{8} \pm 4.3 \cdot 10^{3}$	1
8	$1.10 \cdot 10^{10}$	$1.04 \cdot 10^{9}$	$6.062108 \cdot 10^{6} \pm 3.1 \cdot 10^{2}$	$6.179273 \cdot 10^{9} \pm 1.8 \cdot 10^{5}$	2
9	$1.10 \cdot 10^{10}$	$9.80 \cdot 10^{8}$	$1.045110 \cdot 10^{8} \pm 6.2 \cdot 10^{3}$	$2.692812 \cdot 10^{11} \pm 8.2 \cdot 10^{6}$	2
10	$1.10 \cdot 10^{10}$	$9.33 \cdot 10^{8}$	$1.889201 \cdot 10^{9} \pm 1.3 \cdot 10^{5}$	$1.241497 \cdot 10^{13} \pm 3.9 \cdot 10^{8}$	3
11	$1.10 \cdot 10^{10}$	$8.96 \cdot 10^{8}$	$3.566923 \cdot 10^{10} \pm 2.8 \cdot 10^{6}$	$6.026765 \cdot 10^{14} \pm 1.9 \cdot 10^{10}$	4
12	$1.10 \cdot 10^{10}$	$8.66 \cdot 10^{8}$	$7.012027 \cdot 10^{11} \pm 6.4 \cdot 10^{7}$	$3.071324 \cdot 10^{16} \pm 1.0 \cdot 10^{12}$	5
13	$1.10 \cdot 10^{10}$	$8.44 \cdot 10^{8}$	$1.431902 \cdot 10^{13} \pm 1.5 \cdot 10^{9}$	$1.638982 \cdot 10^{18} \pm 5.4 \cdot 10^{13}$	6
14	$1.10 \cdot 10^{10}$	$8.28 \cdot 10^{8}$	$3.032472 \cdot 10^{14} \pm 3.6 \cdot 10^{10}$	$9.142727 \cdot 10^{19} \pm 3.1 \cdot 10^{15}$	7
15	$3.11 \cdot 10^{11}$	$2.31 \cdot 10^{10}$	$6.655768 \cdot 10^{15} \pm 2.4 \cdot 10^{10}$	$5.323570 \cdot 10^{21} \pm 3.4 \cdot 10^{16}$	249
16	$1.10 \cdot 10^{10}$	$8.10 \cdot 10^{8}$	$1.512467 \cdot 10^{17} \pm 2.4 \cdot 10^{13}$	$3.231993 \cdot 10^{23} \pm 1.1 \cdot 10^{19}$	11
17	$1.10 \cdot 10^{10}$	$8.08 \cdot 10^{8}$	$3.552250 \cdot 10^{18} \pm 6.3 \cdot 10^{14}$	$2.044094 \cdot 10^{25} \pm 6.9 \cdot 10^{20}$	12
18	$1.10 \cdot 10^{10}$	$8.09 \cdot 10^{8}$	$8.632116 \cdot 10^{19} \pm 1.8 \cdot 10^{16}$	$1.345581 \cdot 10^{27} \pm 4.6 \cdot 10^{22}$	15
19	$1.10 \cdot 10^{10}$	$8.12 \cdot 10^{8}$	$2.167796 \cdot 10^{21} \pm 4.9 \cdot 10^{17}$	$9.211519 \cdot 10^{28} \pm 3.1 \cdot 10^{24}$	19
20	$3.00 \cdot 10^{11}$	$2.23 \cdot 10^{10}$	$5.624473 \cdot 10^{22} \pm 4.0 \cdot 10^{17}$	$6.551806 \cdot 10^{30} \pm 4.2 \cdot 10^{25}$	621
25	$8.30 \cdot 10^{10}$	$6.50 \cdot 10^{9}$	$1.066295 \cdot 10^{30} \pm 2.9 \cdot 10^{25}$	$2.060052 \cdot 10^{40} \pm 2.5 \cdot 10^{35}$	824
30	$1.00 \cdot 10^{11}$	$8.26 \cdot 10^{9}$	$4.290822 \cdot 10^{37} \pm 1.8 \cdot 10^{33}$	$1.486361 \cdot 10^{50} \pm 1.6 \cdot 10^{45}$	2032
40	$1.00 \cdot 10^{10}$	$8.86 \cdot 10^{8}$	$4.946806 \cdot 10^{53} \pm 1.6 \cdot 10^{50}$	$6.283492 \cdot 10^{70} \pm 2.0 \cdot 10^{66}$	638
50	$1.00 \cdot 10^{10}$	$9.26 \cdot 10^{8}$	$5.054951 \cdot 10^{70} \pm 3.8 \cdot 10^{67}$	$2.625921 \cdot 10^{92} \pm 8.2 \cdot 10^{87}$	725

Table 2: Primitive β function computation in ϕ^4 theory.

Summary

- Tropical QFT is a, **exactly solvable,** deformation of (scalar) QFT.
- Gives an interesting new toy model for interacting QFTs.
- Geometrically, it is solved by **volume recursions** on the moduli space of graphs/tropical curves.
- Gives a **polynomial-time** algorithm for estimation of scattering amplitudes and other perturbative QFT quantities.
- Solving the **factorial-time** problem of QFT.
- Provides upper and lower bounds on the original QFT.

Open questions

- What is the **non-perturbative** nature of tropical QFT?
- What is the **large loop order** behavior (tropical and non-tropical)?
- **Renormalization** of tropical QFT?

Answers to these questions were recently given: **Balduf—Panzer arXiv:2512.21091**

- **Renormalized** sampling?
- Does the solution of tropical QFT constrain the original? **Bootstrap**?
- Statistical study of e.g. **arithmetic** properties of Feynman integrals.
- Which **non-scalar** theories can be tropicalized? Gauge theory?