Generating functions' asymptotics' generating functions

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- Consider the class of **formal** power series $\mathbb{R}[[x]]^{\alpha}_{\beta} \subset \mathbb{R}[[x]]$ which admit an asymptotic expansion of the form,

$$f_n = \alpha^{n+\beta}\Gamma(n+\beta)\left(c_0 + \frac{c_1}{n} + \frac{c_2}{n(n-1)} + \dots\right)$$

including power series with

$$\lim_{n\to\infty} \frac{f_n}{\alpha^n\Gamma(n+\beta)} = 0$$
$$\Rightarrow c_k = 0 \text{ for all } k \geq 0.$$

- These power series appear in
  - Graph and permutation counting problems in combinatorics.
  - Perturbation expansions in physics.
- Subclass of *gevrey-1*-power series.

- Consider a power series $f(x) \in \mathbb{R}[[x]]^\alpha_\beta$:

$$f_n = \alpha^{n+\beta}\Gamma(n+\beta)\left(c_0 + \frac{c_1}{n} + \frac{c_2}{n(n-1)} + \dots\right)$$

- Idea: Interpret the coefficients c_k of the **asymptotic** expansion as a new power series.

Definition

$\mathcal{A}$ maps a power series to its asymptotic expansion:

$$\begin{array}{cccccc} \mathcal{A} & : & \mathbb{R}[[x]]^\alpha_\beta & \to & \mathbb{R}[[x]] \\ & & f(x) & \mapsto & \gamma(x) = \sum_{k=0}^\infty c_k x^k \end{array}$$

Theorem 1

$\mathcal{A}$ is a derivation on $\mathbb{R}[[x]]^\alpha_\beta$:

$$(\mathcal{A}f \cdot g)(x) = f(x)(\mathcal{A}g)(x) + (\mathcal{A}f)(x)g(x)$$

- Follows from the *log-convexity* of Γ.

$\Rightarrow$ $\mathbb{R}[[x]]^\alpha_\beta$ is a subring of $\mathbb{R}[[x]]$.

Proof sketch

With $h(x) = f(x)g(x)$,

$$h_n = \underbrace{\sum_{k=0}^{R-1} f_{n-k} g_k + \sum_{k=0}^{R-1} f_k g_{n-k}}_{\text{High order times low order}} + \underbrace{\sum_{k=R}^{n-R} f_k g_{n-k}}_{\mathcal{O}(\alpha^n \Gamma(n+\beta-R))}$$

.

Example

- Set $F(x) = \sum_{n=1}^{\infty} n! x^n$,

$$
\begin{aligned}
F \in \mathbb{R}[[x]]_1^1 \quad &\text{and} \quad (\mathcal{A}F)(x) = 1 \\
\Rightarrow F(x)^2 \in \mathbb{R}[[x]]_1^1 & \\
(\mathcal{A}F(x)^2)(x) = F(x)(\mathcal{A}F)(x) + (\mathcal{A}F)(x)F(x) = 2F(x) &
\end{aligned}
$$

- Asymptotic expansion of $F(x)^2$ given by $2F(x)$.

- What happens for **composition** of power series $\in \mathbb{R}[[x]]^\alpha_\beta$?

Theorem 2 Bender [1975]

If $|f_n| \leq C^n$ then, for $g \in \mathbb{R}[[x]]^\alpha_\beta$ with $g_0 = 0$:

$$f \circ g \in \mathbb{R}[[x]]^\alpha_\beta$$
$$(\mathcal{A}f \circ g)(x) = f'(g(x))(\mathcal{A}g)(x)$$

- Bender considered much more general power series, but this is a direct corollary of his theorem in 1975.

Theorem 3 MB [2016]

More general for $f \in \mathbb{R}\{y_1, \ldots, y_L\}$ and $g^1, \ldots, g^L \in \mathbb{R}[[x]]^\alpha_\beta$:

$$(\mathcal{A}(f(g^1(x), \ldots, g^L(x)))(x) = \sum_{l=1}^{L} \frac{\partial f}{\partial y_l}(y_1, \ldots, y_L)\Big|_{\substack{y_m = g^m(x) \\ \forall m \in \{1,\ldots,L\}}} (\mathcal{A}^\alpha_\beta g^l)(x).$$

Example

- Set $F(x) = \sum_{n=1}^{\infty} n! x^n$,

$$
\begin{aligned}
F \in \mathbb{R}[[x]]_1^1 \quad &\text{and} \quad (\mathcal{A}F)(x) = 1 \\
\Rightarrow \cos(F(x)) &\in \mathbb{R}[[x]]_1^1 \\
(\mathcal{A}cos(F(x)))(x) &= -\sin(F(x))(\mathcal{A}F)(x) = -\sin(F(x))
\end{aligned}
$$

- Asymptotic expansion of $\cos(F(x))$ given by $-\sin(F(x))$.

- What happens if $f \notin \ker \mathcal{A}$?
- $\mathcal{A}$ fulfills a general 'chain rule':

Theorem 4 MB [2016]

If $f, g \in \mathbb{R}[[x]]^\alpha_\beta$ with $g_0 = 0$ and $g_1 = 1$:

$$f \circ g \in \mathbb{R}[[x]]^\alpha_\beta$$

$$(\mathcal{A} f \circ g)(x) = f'(g(x))(\mathcal{A} g)(x) + \left(\frac{x}{g(x)}\right)^\beta e^{\frac{g(x)-x}{\alpha x g(x)}} (\mathcal{A} f)(g(x))$$

$\Rightarrow$ We can solve for asymptotics of implicitly defined power series.

- The factor $e^{\frac{g(x)-x}{\alpha x g(x)}}$ generates typical prefactors of the form

$$e^{\frac{g_2}{\alpha}}$$

in asymptotic expansions.

Example: Chord diagrams

- Let $I(x)$ be the ordinary generating function of all chord diagrams and
- $C(x)$ the ordinary generating function of connected chord diagrams.
- They are related by $I(x) = 1 + C(xI(x)^2)$.

$$I(x) = 1 + C(xI(x)^2)$$
$$(\mathcal{A}I)(x) = (\mathcal{A}C(xI(x)^2))(x)$$
$$(\mathcal{A}I)(x) = 2xI(x)C'(xI(x)^2)(\mathcal{A}I)(x) + \left(\frac{x}{xI(x)^2}\right)^{\frac{1}{2}} e^{\frac{xI(x)^2 - x}{2x^2 I(x)^2}} (\mathcal{A}C)(xI(x)^2)$$

$I(x)$ is given by

$$I(x) = \sum_{n=0}^{\infty} (2n-1)!!x^n$$
$$= \sum_{n=0}^{\infty} \frac{2^{n+\frac{1}{2}}}{\sqrt{2\pi}} \Gamma(n + \frac{1}{2}) x^n \in \mathbb{R}[[x]]^2_{\frac{1}{2}}$$

- Using the chain rule for $\mathcal{A}$, we can solve for $(\mathcal{A}C)(x)$:

$$(\mathcal{A}C)(x) = \frac{1}{\sqrt{2\pi}} \frac{x}{C(x)} e^{-\frac{1}{2x}(2C(x)+C(x)^2)}$$

$$(\mathcal{A}C)(x) = \frac{1}{\sqrt{2\pi}} \frac{x}{C(x)} e^{-\frac{1}{2x}(2C(x)+C(x)^2)}$$

$\Rightarrow$ Generating function of the full asymptotic expansion of

$$C_n = (2n-1)!!e^{-1} \left(1 - \frac{5}{2}\frac{1}{2n-1} - \frac{43}{8}\frac{1}{(2n-1)(2n-3)} + \dots \; C_n =\right.$$

Differential equations

- For a **nonlinear ODE** with $F \in \mathbb{R}\{x, y_0, \ldots, y_L\}$ an analytic function and $f \in \mathbb{R}[[x]]^{\alpha}_{\beta}$,

$$0 = F(x, f(x), f'(x), f''(x), \ldots, f^{(L)}(x)).$$

- **Theorem 5 MB [2016]**

$(\mathcal{A}f)(x)$ fulfills the **linear ODE**,

$$0 = \sum_{l=0}^{L} \frac{\partial F}{\partial y_l}(x, y_0, \ldots, y_L)\Big|_{\substack{y_m = f^{(m)}(x) \\ m \in \{0,\ldots,L\}}} (\mathcal{A}f^{(l)})(x),$$

where $(\mathcal{A}f^{(l)})(x) = (\frac{1}{\alpha x^2} - \frac{\beta}{x} + \partial_x)^l (\mathcal{A}f)(x)$.

Example

- Let $f \in \mathbb{R}[[x]]^{\alpha}_{\beta}$ fulfill the ODE,

$$x^2 f'(x) = e^{f(x)} - 1 + x$$

- then

$$(\alpha^{-1} - x\beta + x^2 \partial_x)(\mathcal{A}f)(x) = e^{f(x)}(\mathcal{A}f)(x).$$

- This only has a non-trivial solution if $\alpha = 1$ and $\beta = 1$.
- We obtain $(\mathcal{A}f)(x)$ up to an overall constant.
- $(\mathcal{A}f)(x)$ will depend on initial data of $f(x)$!

Conclusions

- $\mathbb{R}[[x]]^\alpha_\beta$ forms a subring of $\mathbb{R}[[x]]$ **closed under mutliplication, composition, differentiation and integration**.
- $\mathcal{A}$ is a **derivation** on $\mathbb{R}[[x]]^\alpha_\beta$ which can be used to obtain asymptotic expansions of **implicitly defined power series**.
- Nice closure properties under asymptotic derivative $\mathcal{A}$.
- Generalizations possible to multiple $\alpha_1, \ldots, \alpha_l \in \mathbb{C}$ with $|\alpha_i| = \alpha$.
- Suitable for resummation of perturbation series
$\Rightarrow$ applications in QFT and QM!
- There are probably many connections to the theory of resurgence.

Edward A Bender. An asymptotic expansion for the coefficients of some formal power series. *Journal of the London Mathematical Society*, 2(3):451–458, 1975.

MB. Generating asymptotics for factorially divergent sequences. *arXiv preprint arXiv:1603.01236*, 2016.