Graphical functions applied to $\phi^3$ in $D = 6$

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joint work with Oliver Schnetz
Motivation
Objects of interest: Correlation functions

\[ G(x_1, x_2, x_3) \]
Quantum Field theory

- Objects of interest: Correlation functions

\[ G(x_1, x_2, x_3) \]

- Quantifies correlation between points in space.
Objects of interest: Correlation functions

$$G(x_1, x_2, x_3)$$

Quantifies correlation between points in space.

$$G(x_1, x_2, x_3) \in \mathbb{R} \Rightarrow \text{probability of three ‘scalar’ events.}$$
Quantum Field theory

- Objects of interest: **Correlation functions**

\[ G(x_1, x_2, x_3) \]

- Quantifies correlation between points in space.
- \( G(x_1, x_2, x_3) \in \mathbb{R} \Rightarrow \) probability of three ‘scalar’ events.
- \( G(x_1, x_2, x_3) \in \mathbb{V} \Rightarrow \) substructure at each point (e.g. spin).
Quantum Field theory

• Objects of interest: Correlation functions

\[ G(x_1, x_2, x_3) \]

• Quantifies correlation between points in space.
• \( G(x_1, x_2, x_3) \in \mathbb{R} \Rightarrow \) probability of three ‘scalar’ events.
• \( G(x_1, x_2, x_3) \in V \Rightarrow \) substructure at each point (e.g. spin).
• Arbitrary number of points can be correlated \( G(x_1, x_2, x_3, \ldots) \).
Perturbation theory

• No exact formula for correlation functions!
Perturbation theory

- No exact formula for correlation functions!
- We need perturbation theory:

\[ G(x_1, x_2, x_3) = G_0(x_1, x_2, x_3) + \hbar G_1(x_1, x_2, x_3) + \hbar^2 G_2(x_1, x_2, x_3) + \ldots \]
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- Each \( G_n(x_1, x_2, x_3) \) can be written as a sum over graphs:

\[ G_n(x_1, x_2, x_3) = \sum_{\Gamma} \varphi(\Gamma) \]

\[ \chi(\Gamma) = 1 - n \]

The function \( \varphi \) associates an integral to each graph.
Perturbation theory

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The function \( \varphi \) associates an integral to each graph.
- The graphs are called Feynman graphs. The integrals are called Feynman integrals, the function \( \varphi \) is called Feynman rule.
The Feynman integrals are except for the dependence on the physical input algebraic integrals:

\[ \varphi(\Gamma) = \int \frac{d\Omega}{U^{D/2} \left( \frac{U}{F} \right)^{\omega}} \]
Algebraic integrals: Periods

- The Feynman integrals are except for the dependence on the physical input **algebraic integrals**:

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- The **renormalization group independent** part is purely algebraic: The ‘period’

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- There exists various number theoretic conjectures on the period: Coaction conjecture, Cosmic galois group, Motives etc.
Two viewpoints

Momentum space \hspace{2cm} \text{Fourier} \hspace{2cm} \text{Position space}

Correlation functions are parametrized by the momentum of particles

Correlation functions are parametrized by the position of particles
Why position space?
Why position space?

**Advantages**

- Simpler Feynman rules
- No IBP reduction necessary
- Conceptually interesting viewpoint

**Caveats**

- New technology needed
- Only position space quantities accessible

**Proof of concept:**
7-loop \(\beta\)-function in \(\phi^4\) calculated in 2016 by Oliver Schnetz using graphical functions.
Loop integral workflow

Momentum space

Diagram
  Feynman rules

Integral
  Tensor reduction

Scalar integrals
  IBP reduction

Master integrals
  integration

Amplitude
Loop integral workflow

Momentum space

Diagram

Feynman rules

Integral

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Scalar integrals

IBP reduction

Master integrals

integration

Amplitude

annoyingly hard

hard
Loop integral workflow

Momentum space

Diagram → Feynman rules → Integral → Tensor reduction → Scalar integrals → IBP reduction → Master integrals → integration → Amplitude

Position space

Diagram → Graphical reduction → Master diagram → Feynman rules → Integral → Tensor reduction → Scalar integral → integration → Amplitude

*simple*
Loop integral workflow

Momentum space

Diagram
  \rightarrow Feynman rules
  \rightarrow Integral
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  \rightarrow integration
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Position space

Diagram
  \rightarrow Graphical reduction
  \rightarrow Master diagram
  \rightarrow Feynman rules
  \rightarrow Integral
  \rightarrow Tensor reduction
  \rightarrow Scalar integral
  \rightarrow integration
  \rightarrow Amplitude
Feynman integral in momentum space

\[ \tilde{G}(p_1, \ldots, p_n) = \left( \prod_{e \in E} \int d^D k_e \tilde{\Delta}(k_e) \right) \left( \prod_{v \in V_{\text{int}}} \delta^{(D)} \left( \sum_{e \ni v} k_e \right) \right) \]

Lower dimensional integral

Feynman integral in position space

\[ G(x_1, \ldots, x_n) = \left( \prod_{v \in V_{\text{int}}} \int d^D x_v \right) \left( \prod_{\{a, b\} \in E} \Delta(x_a - x_b) \right) \]

Better factorization properties
Examples

**Momentum space**

\[ \tilde{\Delta}(p_{12})\tilde{\Delta}(p_{23})\tilde{\Delta}(p_{31}) \]

\[ \frac{\text{Di}(z,\bar{z})}{\sqrt{-\lambda(p_{12}^2, p_{23}^2, p_{31}^2)}} \]

\[ \tilde{\Delta}(p) = \frac{1}{||p||^2} \]

**Position space**

\[ \Delta(x_{12})\Delta(x_{23})\Delta(x_{31}) \]

\[ \frac{\text{Di}(z,\bar{z})}{\sqrt{-\lambda(x_{12}^2, x_{23}^2, x_{31}^2)}} \]

\[ \Delta(x) = \frac{1}{||x||^2} \]
Graphical reductions
1. rule: propagators between external vertices

\[ G(x_a, x_b, x_c) = \int d^D y \Delta(x_a - y) \Delta(x_b - y) \Delta(x_c - y) \Delta(x_a - x_b) \]
\[ = \Delta(x_a - x_b) H(x_a, x_b, x_c) \]

\[ \Rightarrow \text{edges between external vertices factorize.} \]
2. rule: split graph

⇒ factorizes if split along external vertices.
Intermezzo: amputating a propagator

Recall the definition of the propagator, $\Delta$, as Green’s function for the free field equation

$$(\Box_x - m^2)\Delta(x - y) = \delta^{(D)}(x - y)$$

We can use this equation to amputate free external edges.
Graphical reduction rules

3. rule: amputating an external edge

\((\Box x_a - m^2)G(x_a, x_b, x_c) = \int d^D y (\Box x_a - m^2) \Delta(x_a - y) \Delta(x_b - y) \Delta(x_c - y)\)

\[= \int d^D y \delta(x_a - y) \Delta(x_b - y) \Delta(x_c - y)\]

\[= \Delta(x_b - x_a) \Delta(x_c - x_a) = H(x_a, x_b, x_c)\]

\(\Rightarrow\) solve differential equation to add external edge.
For rule 3, a differential equation needs to be solved:

\[(\Box x_a - m^2) G^{\pi}(x_a, \ldots) = G^{\pi}(x_a, \ldots)\]

Can be solved systematically if (Schnetz 2013)

- particles are massless, \( m = 0 \),
- only 3-point functions are considered
- in \( D = 4 - \epsilon \) Euclidean space.
For rule 3, a differential equation needs to be solved:

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Related approach: (Drummond, Henn, Smirnov 2007) (Magic identities)
3-point configuration space is 2-dimensional, due to Poincare and scaling invariance:

\[ G(x_a, x_b, x_c) = G(x'_a, x'_b, x'_c) \]

for

\[
\begin{align*}
    x'_k \mu &= \Lambda_{\nu}^{\mu} x_k^{\nu} \\
    x'_k \mu &= v^{\mu} + x_k^{\mu}
\end{align*}
\]

with \( \Lambda \in \text{SO}(D) \) and \( v \in \mathbb{R}^D \) and

\[ G(\lambda x_a, \lambda x_b, \lambda x_c) = \lambda^\omega G(x_a, x_b, x_c). \]
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\[ G(\lambda x_a, \lambda x_b, \lambda x_c) = \lambda^\omega G(x_a, x_b, x_c). \]

\( \Rightarrow \) \( G \) only depends on the shape of the triangle spanned by \( x_a, x_b, x_c. \)
Exploit this symmetry by using complex parameter $z$ such that

$$z \bar{z} = \frac{x_{ac}^2}{x_{ab}^2} \quad \text{and} \quad (1 - z)(1 - \bar{z}) = \frac{x_{bc}^2}{x_{ab}^2}$$
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$$\Box_{x_c} (x_a, x_b, x_c) = G (x_a, x_b, x_c)$$

$$\frac{1}{z - \bar{z}} \partial_z \partial_{\bar{z}} (z - \bar{z}) \quad G (z, \bar{z}) = G (z, \bar{z})$$

The $\partial_z$ and $\partial_{\bar{z}}$ operators can be inverted in the function space of generalized single-valued hyperlogarithms (Chavez, Duhr 2012, Schnetz 2014, Schnetz 2017).
Graphical functions

- Rules 1, 2, 3 are part of a larger framework: graphical functions (Schnetz 2013).
- Graphical functions can also be applied in a broader context, e.g. to conformal amplitudes (Basso, Dixon 2017).
- Calculation within this framework are extremely efficient, due to the rapid reductions and small numbers of irreducible master diagrams.
- Additional identities specific to the theory (e.g. conformal transformations for scalar theories).
Graphical functions for gauge theory
Beyond scalar

Only change: adding an edge

For instance, for abelian gauge theory:

\[ \Box_x \rightarrow \partial \text{ and } \eta_{\mu\nu} \Box_x \]
**Only change: adding an edge**

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\[ \Box_x \rightarrow \partial \text{ and } \eta^{\mu\nu} \Box_x \]

The differential equation for appending an edge,

\[ \Box_{x_a} G(\ldots) = G(\ldots) \]

becomes a system of differential equations

\[ \partial_{x_a} G(\ldots) = G(\ldots) \]
Parametrizing non-scalar graphical functions

\[ \hat{\phi}_{x_c} \quad G(x_a, x_b, x_c) = \hat{G}(x_a, x_b, x_c) \]
Parametrizing non-scalar graphical functions

\[
\frac{\partial}{\partial x_c} G(x_a, x_b, x_c) = G(x_a, x_b, x_c)
\]

\[
\left( \frac{\lambda}{z} \frac{\partial}{\partial z} + \frac{\bar{\lambda}}{\bar{z}} \frac{\partial}{\partial \bar{z}} - \frac{P^{\mu\nu}}{z - \bar{z}} \left( \frac{\partial}{\partial x^\mu} - \frac{\partial}{\partial x^\nu} \right) \right) G(z, \bar{z}, \lambda, \bar{\lambda}) = G(z, \bar{z}, \lambda, \bar{\lambda})
\]

Using light-cone-like parametrization \( z, \bar{z}, \lambda^\mu, \bar{\lambda}^\mu \) such that

\[
z \bar{z} = \frac{x_{ac}^2}{x_{ab}^2} \quad \text{and} \quad (1 - z)(1 - \bar{z}) = \frac{x_{bc}^2}{x_{ab}^2}
\]

\[
x_{ab}^{\mu} = \lambda^{\mu} + \bar{\lambda}^{\mu} \quad x_{ac}^{\mu} = z \lambda^{\mu} + \bar{z} \bar{\lambda}^{\mu} \quad x_{bc}^{\mu} = (1 - z) \lambda^{\mu} + (1 - \bar{z}) \bar{\lambda}^{\mu}
\]

\[
\lambda^{\mu} \lambda_{\mu} = \bar{\lambda}^{\mu} \bar{\lambda}_{\mu} = 0
\]

Actual inversion becomes more complicated: \( D \neq 4 \) dimensional Laplacian has to be inverted.
Diagonalization of the equation system gives,

\[
\begin{pmatrix}
\Delta_D & 0 & 0 \\
0 & \Delta_{D+2} & 0 \\
0 & 0 & \Delta_{D+4}
\end{pmatrix}
\sim
\tilde{G}(x_a, x_b, x_c) = \tilde{G}(x_a, x_b, x_c),
\]

where \( \Delta_D = \frac{2}{z - \bar{z}} \partial_z \partial_{\bar{z}} (z - \bar{z}) - \frac{D-4}{z - \bar{z}} (\partial_z - \partial_{\bar{z}}). \)
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where \( \Delta_D = \frac{2}{z - \bar{z}} \partial_z \partial_{\bar{z}} (z - \bar{z}) - \frac{D-4}{z - \bar{z}} (\partial_z - \partial_{\bar{z}}) \).

\Rightarrow \text{we would like to invert } \Delta_D \text{ for general even } D.
Extension to $D \neq 4$

- For general dimension $D$ we need to solve,

$\left( \frac{2}{z - \bar{z}} \partial_z \partial_{\bar{z}} (z - \bar{z}) - \frac{D - 4}{z - \bar{z}} (\partial_z - \partial_{\bar{z}}) \right) G (z, \bar{z}) = G (z, \bar{z})$. 
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- This is also possible for arbitrary even $D$ using a non-trivial linear combination of integration operators.
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⇒ Opens the door to calculations in gauge theories.

⇒ Immediately possible tools: $\phi^3$-theory. With applications to percolation theory and other variants (e.g. biadjoint $\phi^3$).
An inverse to the differential operator

\[ \frac{1}{2} \Delta_{2+2n} = \frac{1}{z - \bar{z}} \partial_z \partial_{\bar{z}} (z - \bar{z}) - \frac{n-1}{z - \bar{z}} (\partial_z - \partial_{\bar{z}}) \]

is given by the integration operator:

\[ I_n = \sum_{k,l=0}^{n} c_{n,k,l} (z - \bar{z})^{-k} \int_{SV} d\, z (z - \bar{z})^{k+l} \int_{SV} d\, \bar{z} (z - \bar{z})^{-l} \]

where \( c_{n,k,l} \) are some easily determined coefficients.
\[ \beta_{\phi^3}(g) = \left( \frac{5}{2016} \pi^6 - \frac{46519}{829440} \pi^4 + \frac{102052031}{6718464} \zeta(3)^2 + \frac{99}{16} \zeta(3)^2 \right) g^{11} + \]
\[ + \left( \frac{366647}{6912} \zeta(3) + \frac{151795}{3456} \zeta(5) - \frac{102052031}{6718464} \zeta(7) \right) g^{11} + \]
\[ + \left( \frac{1}{192} \pi^4 - \frac{3404365}{746496} \zeta(3) + \frac{5}{3} \zeta(5) \right) g^9 + \]
\[ + \left( \frac{33085}{20736} + \frac{5}{8} \zeta(3) \right) g^7 - \frac{125}{144} g^5 + \frac{3}{4} g^3 \]

4- and 3-loop results due to (John Gracey 2015; de Alcantara Bonfim, Kirkham, McKane, 1980).

⇒ More accurate predictions for the critical exponents in percolation theory and for the Lee-Yang edge singularity.
Summary

- Efficient \textit{graphical reduction} replaces IBP reduction in $x$-space.
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• Application of $\phi^3$-theory: Critical exponents in percolation theory.
Summary

- Efficient graphical reduction replaces IBP reduction in x-space.
- Work in progress: extension to gauge theory.
- Intermediate step finished: extension to arbitrary even \( D \).
- Application of \( \phi^3 \)-theory: Critical exponents in percolation theory.
- Question: Extension to odd \( D \) possible?
Example of a master diagram, which is irreducible w.r.t. rules 1–3: