Counting graphs without given edge-induced subgraphs

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Combinatorial structures in perturbative QFT 2:
transforms and graph counting
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Motivation

Why study graphs with forbidden subgraphs?

- Applications to perturbative QFT especially combined with asymptotic techniques.
- Applications to complex systems, network theory and random graphs.
- Interesting combinatorial and algebraic properties.

Generating functions of graphs

For the set of all multigraphs $\mathfrak{K} = \mathfrak{M}$, we have the explicit formula,

$$f_{\mathfrak{M}}(\lambda_0, \lambda_1, \lambda_2, \ldots) = \sum_{\Gamma \in \mathfrak{M}} \frac{\prod_{v \in V_\Gamma} \lambda_{d_\Gamma^{(v)}}}{|\operatorname{Aut} \Gamma|}$$

$$= \sum_{n \geq 0} (2n-1)!![x^{2n}] e^{\sum_{k \geq 0} \lambda_k \frac{x^k}{k!}}$$

where the λ_k **mark** the number of vertices of degree k.

- Combinatorial interpretation:
  graph = perfect matching x set partition on half edges
- This is also the starting point for zero-dimensional QFT and the pairing model of multigraphs. Hurst [1952], Cvitanović, Lautrup, and Pearson [1978], Bender and Canfield [1978], Argyres, van Hameren, Kleiss, and Papadopoulos [2001]

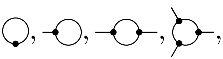
Restrictions on the allowed subgraphs

- Would like to obtain a generating function

$$f_{\mathfrak{M}/\mathfrak{P}}(\lambda_0, \lambda_1, \ldots) := \sum_{\substack{\Gamma \in \mathfrak{M} \\ \text{s.t. } \Gamma \text{ has no subgraphs from } \mathfrak{P}}} \frac{\prod_{v \in V_\Gamma} \lambda_{d_\Gamma^{(v)}}}{|\operatorname{Aut} \Gamma|}$$

- Graphs in $\mathfrak{P}$ shall keep the same degree as in the original graph.

  This can be done be including graphs with **legs**. For instance,

$\mathfrak{P} = \{$  $, \ldots\}$

The inverse problem

- Suppose, we have a generating function

$$f_{\mathfrak{M}/\mathfrak{P}}(\lambda_0, \lambda_1, \ldots) = \sum_{\substack{\Gamma \in \mathfrak{M} \\ \text{s.t. } \Gamma \text{ has no subgraphs from } \mathfrak{P}}} \frac{\prod_{v \in V_\Gamma} \lambda_{d_\Gamma^{(v)}}}{|\operatorname{Aut} \Gamma|}$$

- and the generating functions of the graphs in $\mathfrak{P}$:

$$f_{\mathfrak{P}}^{k}(\lambda_0, \lambda_1, \ldots) = \sum_{\substack{\Gamma \in \mathfrak{P} \\ \text{s.t. } \Gamma \text{ is cntd. and has } k \text{ legs}}} \frac{\prod_{v \in V_\Gamma} \lambda_{d_\Gamma^{(v)}}}{|\operatorname{Aut} \Gamma|}$$

- Can we reconstruct $f_{\mathfrak{M}}$, the generating function of all graphs?

Theorem MB [2018 PhD thesis]

If for all $\gamma_1, \gamma_2 \subset \Gamma \in \mathfrak{M}$, $\gamma_1, \gamma_2 \in \mathfrak{P}$ implies $\gamma_1 \cup \gamma_2 \in \mathfrak{P}$, then

$$f_{\mathfrak{M}}(\lambda_0, \lambda_1, \ldots) = f_{\mathfrak{M}/\mathfrak{P}}\left((0!) f_{\mathfrak{P}}^0(\lambda_0, \ldots), (1!) f_{\mathfrak{P}}^1(\lambda_0, \ldots), \ldots\right)$$

- In words: The union of any pair of forbidden **subgraphs** is also forbidden.

Example

- Suppose $\mathfrak{P}_{\text{bl}}$ is the set of all bridgeless graphs.
- Clearly, for any two bridgeless subgraphs $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \cup \gamma_2$ is also bridgeless.
- Therefore,

$$f_{\mathfrak{M}}(\lambda_0, \lambda_1, \dots) = f_{\mathfrak{M}/\mathfrak{P}_{\text{bl}}}\left((0!) f^0_{\mathfrak{P}_{\text{bl}}}(\lambda_0, \dots), (1!) f^1_{\mathfrak{P}_{\text{bl}}}(\lambda_0, \dots), \dots\right)$$

- $\mathfrak{M}/\mathfrak{P}_{\text{bl}}$ is the set of all graphs without bridgeless subgraphs.

  I.e. forests.

The generating function of forests is

$$f_{\mathfrak{M}/\mathfrak{P}_{\text{bl}}}(\lambda_0, \lambda_1, \ldots) = \sum_{\substack{\Gamma \in \mathfrak{M} \\ \text{s.t. } \Gamma \text{ is a forest}}} \frac{\prod_{v \in V_\Gamma} \lambda_{d_\Gamma^{(v)}}}{|\operatorname{Aut}\Gamma|} = e^{-\frac{\varphi_c^2}{2} + \sum_{k\geq 0} \lambda_k \frac{\varphi_c^k}{k!}}$$

where $\varphi_c \in \mathbb{Q}[[\lambda_1, \lambda_2, \ldots]]$, the unique solution of

$$\varphi_c = \sum_{k\geq 0} \lambda_{k+1} \frac{\varphi_c^k}{k!},$$

the generating function of **rooted trees**.

$$f_{\mathfrak{M}}(\lambda_0, \lambda_1, \ldots) = f_{\mathfrak{M}/\mathfrak{P}_{\text{bl}}}\left((0!) f^0_{\mathfrak{P}_{\text{bl}}}(\lambda_0, \ldots), (1!) f^1_{\mathfrak{P}_{\text{bl}}}(\lambda_0, \ldots), \ldots\right)$$

- Using the generating function of forests we get,

$$f_{\mathfrak{M}}(\lambda_0, \lambda_1, \ldots) = e^{-\frac{\widetilde{\varphi}_c^2}{2} + \sum_{k\geq 0} f^k_{\mathfrak{P}_{\text{bl}}}(\lambda_0, \lambda_1, \ldots)\widetilde{\varphi}_c^k}$$

  where $\widetilde{\varphi}_c$ is the solution of $\widetilde{\varphi}_c = \sum_{k\geq 0} k f^{k+1}_{\mathfrak{P}_{\text{bl}}}(\lambda_0, \lambda_1, \ldots)\widetilde{\varphi}_c^k$
- Only one graph is bridgeless and has a one-valent vertex:  , the vertex with one leg.
- Therefore,

$$\frac{\partial}{\partial \lambda_1} \log\left(f_{\mathfrak{M}}(\lambda_0, \lambda_1, \ldots)\right) = \frac{\partial}{\partial \lambda_1}\left(-\frac{\widetilde{\varphi}_c^2}{2} + \sum_{k\geq 0} f^k_{\mathfrak{P}_{\text{bl}}}(\lambda_0, \lambda_1, \ldots)\widetilde{\varphi}_c^k\right)$$

$$= \frac{\partial \widetilde{\varphi}_c}{\partial \lambda_1}\frac{\partial}{\partial \widetilde{\varphi}_c}\left(-\frac{\widetilde{\varphi}_c^2}{2} + \sum_{k\geq 0} f^k_{\mathfrak{P}_{\text{bl}}}(\lambda_0, \lambda_1, \ldots)\widetilde{\varphi}_c^k\right) + \sum_{k\geq 0}\frac{\partial f^k_{\mathfrak{P}_{\text{bl}}}}{\partial \lambda_1}\widetilde{\varphi}_c^k$$

$$= \widetilde{\varphi}_c$$

- Gives an expression for the generating functions $f^k_{\mathfrak{P}_{\text{bl}}}(\lambda_0, \lambda_1, \ldots)$ of connected bridgeless graphs:

$$\sum_{k \geq 0} f^k_{\mathfrak{P}_{\text{bl}}}(\lambda_0, \lambda_1, \ldots) \widetilde{\varphi}_c^k = \frac{\widetilde{\varphi}_c^2}{2} + \log\left(f_{\mathfrak{M}}(\lambda_0, \lambda_1, \ldots)\right)$$

where we only need to solve the equation,

$$\widetilde{\varphi}_c = \frac{\partial}{\partial \lambda_1} \log\left(f_{\mathfrak{M}}(\lambda_0, \lambda_1, \ldots)\right)$$

for λ_1 to obtain $f^k_{\mathfrak{P}_{\text{bl}}}(\lambda_0, \lambda_1, \ldots)$ explicitly. This operation is a **Legendre transformation**.

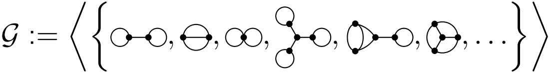
- We are able to reduce the equation,

$$f_{\mathfrak{M}}(\lambda_0, \lambda_1, \dots) = f_{\mathfrak{M}/\mathfrak{P}_{\text{bl}}}\left((0!) f^0_{\mathfrak{P}_{\text{bl}}}(\lambda_0, \dots), (1!) f^1_{\mathfrak{P}_{\text{bl}}}(\lambda_0, \dots), \dots\right)$$

  which is implicit in an **infinite** number of variables to a **single** implicit equation.
- To study the reason for this an algebraic viewpoint is helpful.

Algebra homomorphisms of graphs

Define an algebra generated by all graphs with disjoint union as multiplication,



- We can study algebra homomorphisms $\phi : \mathcal{G} \to \mathbb{A}$.

- For instance,

$$\phi : \mathcal{G} \to \mathbb{Q}[[\lambda_0, \lambda_1, \ldots]]$$
$$\phi : \Gamma \mapsto \prod_{v \in V_\Gamma} \lambda_{d_\Gamma^{(v)}}$$

- Setting $\mathfrak{X} = \sum_{\Gamma \in \mathfrak{M}} \frac{\Gamma}{|\operatorname{Aut}\Gamma|}$ we recover,

$$\phi(\mathfrak{X}) = f_{\mathfrak{M}}(\lambda_0, \lambda_1, \ldots) = \sum_{n \geq 0} (2n-1)!![x^{2n}] e^{\sum_{k \geq 0} \lambda_k \frac{x^k}{k!}}$$

- We would like to put compositions of the form

$$f\left(g^0(\lambda_0,\ldots), g^1(\lambda_0,\ldots),\ldots\right)$$

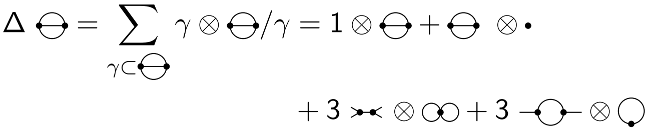
  into the algebraic framework.
- To do this we use a generalization of the Connes-Kreimer Hopf algebra Connes and Kreimer [2001].

- The starting point is to equip $\mathcal{G}$ with a **coproduct**:

$$\begin{aligned} \Delta : \quad & \mathcal{G} & \to & \quad \mathcal{G} \otimes \mathcal{G} \\ & \Gamma & \mapsto & \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma \end{aligned}$$

where the sum is over all edge-induced subgraphs of Γ.

Example:



- The coproduct gives rise to a **group structure** $\Phi^{\mathcal{G}}_{\mathbb{A}}$ on the set of all algebra homomorphisms.
- If ϕ and ψ are algebra homomorphisms $\mathcal{G} \to \mathbb{A}$, then

$$\phi \star \psi = m \circ (\phi \otimes \psi) \circ \Delta$$

is another algebra homomorphism.

- We have an identity on $\mathcal{G}$ Kreimer [2006], Yeats [2008], MB [2014]

$$\Delta \mathfrak{X} = \sum_{\Gamma} \prod_{v \in V_\Gamma} (d_\Gamma^{(v)}!) \mathfrak{X}^{(v)} \otimes \frac{\Gamma}{|\operatorname{Aut} \Gamma|},$$

  where $\mathfrak{X} = \sum_\Gamma \frac{\Gamma}{|\operatorname{Aut}\Gamma|}$ and $\mathfrak{X}^{(v)} := \sum_{\operatorname{res}\Gamma = v} \frac{\Gamma}{|\operatorname{Aut}\Gamma|}$.
- Allows the explicit evaluation of products of algebra homomorphisms.
- For instance if $\psi, \zeta \in \Phi_{\mathbb{A}}^{\mathcal{G}}$ with ζ being the characteristic map $\zeta : \Gamma \mapsto 1$,

$$\begin{aligned} \psi \star \zeta (\mathfrak{X}) &= \sum_{\Gamma \in \mathfrak{M}} \prod_{v \in V_\Gamma} \frac{(d_\Gamma^{(v)}!) \psi \left(\mathfrak{X}^{(v)} \right)}{|\operatorname{Aut}\Gamma|} \\ &= f_{\mathfrak{M}} \left((0!) \psi \left(\mathfrak{X}^{(0)} \right), (1!) \psi \left(\mathfrak{X}^{(1)} \right), \ldots \right) \end{aligned}$$

  where we recover our generalized composition of power series.

Hopf ideals in $\mathcal{G}$ MB [2018 PhD thesis]

A given set of graphs $\mathfrak{P}$, which is **closed under insertion and contraction of subgraphs** corresponds to a **Hopf ideal** $I_{\mathfrak{P}}$ of $\mathcal{G}$.

- Every ideal $I_{\mathfrak{P}}$ gives rise to another group $\Phi_{\mathbb{A}}^{\mathcal{G}/I_{\mathfrak{P}}}$ which **acts** on $\Phi_{\mathbb{A}}^{\mathcal{G}}$.
- Quotients $\mathcal{G}/I_{\mathfrak{P}}$ give rise to annihilation mappings,

Theorem MB [2018 PhD thesis]

$$\zeta^{\star -1}|_{\mathfrak{P}} \star \zeta(\Gamma) = \begin{cases} 1 & \text{if } \Gamma \text{ does not contain a subgraph from } \mathfrak{P}. \\ 0 & \text{else} \end{cases}$$

where ζ is the characteristic map $\zeta : \Gamma \mapsto 1$.

- These maps allow us to obtain generating functions of graphs without subgraphs in $\mathfrak{P}$.

By using the Theorem and the factorization formula for the coproduct:

$$sk_{\boldsymbol{\lambda}} \star (\zeta^{\star -1}|_{\mathfrak{P}} \star \zeta)(\mathfrak{X}) = \sum_{\substack{\Gamma \in \mathfrak{M} \\ \text{s.t. } \Gamma \text{ has no subgraphs from } \mathfrak{P}}} \frac{\prod_{v \in V_\Gamma} \lambda_{d_\Gamma^{(v)}}}{|\operatorname{Aut}\Gamma|}$$

$$(sk_{\boldsymbol{\lambda}} \star \zeta^{\star -1}|_{\mathfrak{P}}) \star \zeta(\mathfrak{X}) = \sum_{\Gamma \in \mathfrak{M}} \prod_{v \in V_\Gamma} \frac{(d_\Gamma^{(v)}!) sk_{\boldsymbol{\lambda}} \star \zeta^{\star -1}|_{\mathfrak{P}} \left(\mathfrak{X}^{(v)}\right)}{|\operatorname{Aut}\Gamma|}$$

$$= f_{\mathfrak{M}} \left((0!) sk_{\boldsymbol{\lambda}} \star \zeta^{\star -1}|_{\mathfrak{P}} \left(\mathfrak{X}^{(0)}\right), (1!) sk_{\boldsymbol{\lambda}} \star \zeta^{\star -1}|_{\mathfrak{P}} \left(\mathfrak{X}^{(1)}\right), \ldots \right)$$

where $sk_{\boldsymbol{\lambda}}$ maps graphs without edges to $\prod_{v \in V_\Gamma} \lambda_{d_\Gamma^{(v)}}$ and all other graphs to 0.

Therefore,

$$f_{\mathfrak{M}}(\lambda_0, \lambda_1, \dots) = f_{\mathfrak{M}/\mathfrak{P}}\left((0!)f^0_{\mathfrak{P}}(\lambda_0, \dots), (1!)f^1_{\mathfrak{P}}(\lambda_0, \dots), \dots\right)$$

can be solved for the generating function of graphs without subgraphs in $\mathfrak{P}$:

$$f_{\mathfrak{M}/\mathfrak{P}}(\lambda_0, \lambda_1, \dots) = f_{\mathfrak{M}}\left((0!)g^0_{\mathfrak{P}}(\lambda_0, \dots), (1!)g^1_{\mathfrak{P}}(\lambda_0, \dots), \dots\right)$$

where

$$\begin{aligned} g^k_{\mathfrak{P}}(\lambda_0, \lambda_1, \dots) &= sk_{\boldsymbol{\lambda}} \star \zeta^{\star -1}|_{\mathfrak{P}}\left(\mathfrak{X}^{(k)}\right) \\ &= \sum_{\substack{\Gamma \in \mathfrak{P} \\ \Gamma \text{ cntd. with } k \text{ legs}}} \zeta^{\star -1}|_{\mathfrak{P}}(\Gamma) \frac{\prod_{v \in V_\Gamma} \lambda_{d^{(v)}_\Gamma}}{|\operatorname{Aut} \Gamma|} \end{aligned}$$

and $\zeta^{\star -1}|_{\mathfrak{P}}(\Gamma)$ can be expressed as a Moebius function,

$$\zeta^{\star -1}|_{\mathfrak{P}}(\Gamma) = -1 - \sum_{\substack{\gamma \subsetneq \Gamma \\ \gamma \in \mathfrak{P}}} \zeta^{\star -1}|_{\mathfrak{P}}(\gamma)$$

Example

- Set $\mathfrak{P}_{-\bullet}$ to the set of all graphs with one leg, for instance .
- Clearly, this set is closed under contraction and insertion of subgraphs.
- The set $\mathfrak{M}/\mathfrak{P}_{-\bullet}$ of graphs without subgraphs from $\mathfrak{P}_{-\bullet}$ is the set of bridgeless graphs $\Rightarrow$ dual to our first problem!
- Using our results,

$$f_{\mathfrak{M}/\mathfrak{P}_{-\bullet}}(\lambda_0, \lambda_1, \ldots) = f_{\mathfrak{M}}\left((0!) g^0_{\mathfrak{P}_{-\bullet}}(\lambda_0, \ldots), (1!) g^1_{\mathfrak{P}_{-\bullet}}(\lambda_0, \ldots), \ldots \right)$$

  where now $g^k_{\mathfrak{P}_{-\bullet}}(\lambda_0, \ldots) = \lambda_k$ for all $k \neq 1$.
- Moreover, by analysing the Moebius function we find that

$$g^1_{\mathfrak{P}_{-\bullet}}(\lambda_0, \ldots) = - \sum_{\substack{\Gamma \in \mathfrak{P}_{-\bullet} \\ \text{s.t. } \Gamma \text{ is bridgeless}}} \frac{\prod_{v \in V_\Gamma} \lambda_{d_\Gamma^{(v)}}}{|\operatorname{Aut} \Gamma|}$$

Conclusions

- Under certain conditions, we may obtain the generating functions of graphs without a given set of subgraphs.
- Explicit formulas may be obtained using Hopf algebra techniques.
- We can also mark instances of subgraphs with extra variables using specific algebra homomorphisms.
- Asymptotics are accessible using techniques in the ring of factorially divergent power series MB [2016].

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